VECTOR CALCULUS

IIT-JAM, TIFR, IISc, DU, BHU, ISI

CMI, BSc (H), IAS etc.

First Edition

Vikas Deoarshi

B.Tech, IIT-BHU

Vishal Deoarshi

B.Tech, IIT-BHU

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Head Office : 89, Mall Road, Kingsway Camp, Delhi-09

Phone : 011-45644998, Fax : 011-45058016, Email : info@samvedna.com

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PREFACE

The continued demand of the book on **VECTOR CALCULUS** from student community has prompted us to write a book which can serve their needs. This book will go a long way in helping students in their graduate course and in preparation of M.Sc. Entrance exams like **IIT-JAM, TIFR, IISc, CMI,ISI, DU, BHU, ISM** and Civil Services Exams.

The questions of previous year papers of **different competitive exams** been solved as if discussed in class by a teacher. The book has been divided into ten chapters.

We give special thanks to all my colleagues at **TRAJECTORY EDUCATION** with special reference to Mrs. Jyoti, Mr. Amit, Mr Rajneesh, Ms Ravneet, and **SAMVEDNA PUBLICATION** team with special reference to Ms. Dashmeet Kaur, Mr. Shankar Choudhary, Mr. Ajeet Kumar, Mr. Devid, Mr. Kishan Kumar, Mr. Ravi Kumar, Mr. Sanjeev Kumar, Ms. Rakhi, Ms. Neetu who always supported us while writing this book.

We also thank Dr. P.K.Chakraborty, Head, Department of Mathematics, MJK College, Bettiah, Bihar for his valuable inputs.

Finally, we thank all our students and my family with special reference to my (vishal) daughter whose appreciation and love constantly motivated us.

We would be gratified to receive the comments, critical observation and suggestions from the readers. These will be incorporated in the subsequent editions

VIKAS DEOARSHI

VISHAL DEOARSHI

CONTENT

1

VECTOR ALGEBRA

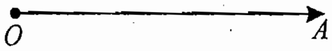
1. DEFINITION

A scalar is a quantity, which has only magnitude but does not have a direction. For example: time, mass, temperature, distance and specific gravity etc. are scalars.

A Vector is a quantity which has magnitude, direction and follow the triangle law of addition. For example : displacement, force, acceleration are vectors.

(a) There are different ways of denoting a vector: $\vec{a}$ or $\bar{a}$ or a are different ways. We use for our convenience $\vec{a}, \vec{b}, \vec{c}$ etc. to denote vectors, and a, b, c to denote their magnitude. Magnitude of a vector $\vec{a}$ is also written as $|\vec{a}|$.

(b) A vector $\vec{a}$ may be represented by a line segment OA and arrow gives direction of this vector. Length of the line segment gives the magnitude of the vector.



Here O is the initial point and A is the terminal point of OA

Figure 1.1

2. ADDITION OF TWO VECTORS

Let $\overline{OA} = \vec{a}$, $\overline{AB} = \vec{b}$ and $\overline{OB} = \vec{c}$.

Here $\vec{c}$ is sum (or resultant) of vectors $\vec{a}$ and $\vec{b}$. It is to be noticed that the initial point of $\vec{b}$ coincides with the terminal point of $\vec{a}$ and the line joining the initial point of $\vec{a}$ to the terminal point of $\vec{b}$ represents vector $\vec{a}+\vec{b}$ in magnitude and direction.

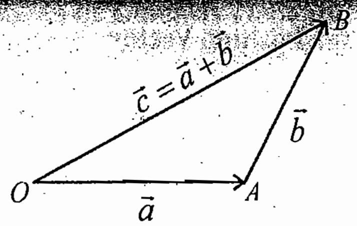


Figure 1.2

2.1 Properties

(i) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$, (Vector addition is commutative)

(ii) $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$, (Vector addition is associative)

(iii) $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$, equality holds when $\vec{a}$ and $\vec{b}$ are like vectors

(iv) $|\vec{a}+\vec{b}| \ge ||\vec{a}|-|\vec{b}||$, equality holds when $\vec{a}$ and $\vec{b}$ are unlike vectors

(v) $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$.

(vi) $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$

3. TYPES OF VECTORS

(i) Equal Vectors

Two vectors are said to be equal if and only if they have equal magnitudes and same direction.

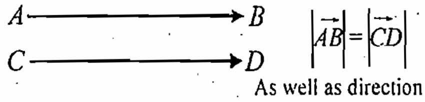


Figure 1.3

(ii) Zero Vector (Null Vector)

A vector whose initial and terminal points are same, is called the null vector. For example $\overline{AA}$. Such vector has zero magnitude and no direction, and denoted by $\vec{0}$.

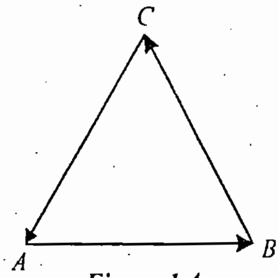


Figure 1.4

$\overline{AB}+\overline{BC}+\overline{CA}=\overline{AA}$ or $\overline{AB}+\overline{BC}+\overline{CA}=\vec{0}$

(iii) Like and Unlike Vectors

Two vectors are said to be

(a) Like, when they have same direction.

(b) Unlike, when they are in opposite directions $\vec{a}$ and $-\vec{a}$ are two unlike vectors as their directions are opposite, $\vec{a}$ and $3\,\vec{a}$ are like vectors.

(iv) Unit Vector

A unit vector is a vector whose magnitude is unity. We write, unit vector in the direction of $\vec{a}$ as $\hat{a}$.

Therefore $\hat{a}=\dfrac{\vec{a}}{|\vec{a}|}$

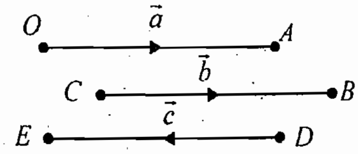


Figure 1.5

(v) Parallel Vectors

Two or more vectors are said to be parallel, if they have the same support or parallel support. Parallel vectors may have equal or unequal magnitudes and direction may be same or opposite. As shown in figure

(vi) Position Vector

If P is any point in the space then the vector $\overline{OP}$ is called position vector of point P, where O is the origin of reference. Thus for any points A and B in the space, $\overline{AB}=\overline{OB}-\overline{OA}$

(vii) Co-initial vectors

Vectors having same initial point are called co-initial vectors. As shown in figure: Here $\overline{OA}, \overline{OB}, \overline{OC}$ and $\overline{OD}$ are co-initial vectors.

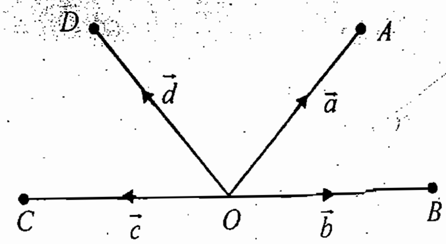


Figure 1.6

4. MULTIPLICATION OF VECTOR BY SCALARS

If $\vec{a}$ is a vector and m is a scalar, then $m\,\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called ***Scalar Multiplication***. If $\vec{a}$ and $\vec{b}$ are vectors and m, n are scalars, then :

$$m(\vec{a})=(\vec{a})m=m\vec{a} \qquad m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$$

$$(m+n)\vec{a}=m\vec{a}+n\vec{a} \qquad m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$$

1. If $\vec{a}$ and $\vec{b}$ are the vectors determined by two adjacent sides of a regular hexagon, what are the vectors determined by the other sides taken in order ?

Solution.

$OABCDE$ is a regular hexagon. Let $\overline{OA} = \vec{a}$ and $\overline{AB} = \vec{b}$. Join OB and OC

We have

$$\overline{OB} = \overline{OA} + \overline{AB} = \vec{a} + \vec{b}$$

Since, OC is parallel to AB and double of AB.

$$\therefore \quad \overline{OC} = 2\overline{AB} = 2\vec{b}$$

Now $\quad \overline{BC} = \overline{OC} - \overline{OB} = 2\vec{b} - (\vec{a} + \vec{b}) = \vec{b} - \vec{a}$

$\overline{CD} = -\overline{OA} = -\vec{a}$ and $\overline{DE} = -\overline{AB} = -\vec{b}$

Also $\quad \overline{EO} = -\overline{BC} = -(\vec{b} - \vec{a}) = \vec{a} - \vec{b}$

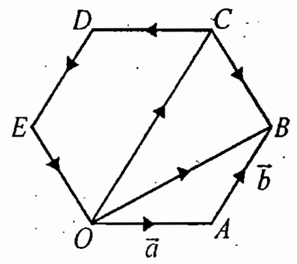


Figure 1.7

5. LINEAR COMBINATIONS

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots$ then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, \ldots$ for any $x, y, z \ldots \in R$. We have the following results :

(i) If $\vec{a}, \vec{b}$ are non zero, non-collinear vectors then $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'; y = y'$

(ii) **Fundamental theorem:** Let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. Then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ *i.e.* There exist some unique $x, y \in R$ such that $x\vec{a} + y\vec{b} = \vec{r}$.

(iii) If $\vec{a}, \vec{b}, \vec{c}$ are non-zero, non-coplanar vectors then:

$$x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c} \quad \Rightarrow \quad x = x', \ y = y', \ z = z'$$

(iv) **Fundamental theorem in space:** Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ *i.e.* There exist some unique $x, y \in R$ such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

(v) If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are n non zero vectors, & $k_1, k_2, \ldots k_n$ are n scalars & if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + \cdots k_n\vec{x}_n = 0 \Rightarrow k_1 = 0, \ k_2 = 0 \ldots k_n = 0$ then we say that vectors $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_n$ are **Linearly Independent Vectors.**

(vi) If $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_n$ are not **Linearly Independent** then they are said to be **Linearly Dependent** vectors. *i.e.* if $k_1\vec{x}_1 + k_2\vec{x}_2 + \ldots\ldots + k_n\vec{x}_n = 0$ & if there exists at least one $k_r \neq 0$ then $\vec{x}_1, \vec{x}_2, \ldots\ldots \vec{x}_n$ are said to be **Linearly Dependent.**

Note 1:

If $k_r \neq 0$; $\quad k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 + \cdots + k_r\vec{x}_r + \cdots + k_n\vec{x}_n = 0$

$$-k_r\vec{x}_r = k_1\vec{x}_1 + k_2\vec{x}_2 + \cdots + k_{r-1}.\vec{x}_{r-1} + k_{r+1}\cdot\vec{x}_{r+1} + \cdots + k_n\vec{x}_n$$

$$-k_r\frac{1}{k_r}\vec{x}_r = k_1\frac{1}{k_r}\vec{x}_1 + k_2\frac{1}{k_r}\vec{x}_2 + \cdots + k_{r-1}\cdot\frac{1}{k_r}\vec{x}_{r-1} + \cdots + k_n\frac{1}{k_r}\vec{x}_n$$

$$\vec{x}_r = c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_{r-1}\vec{x}_{r-1} + c_r\vec{x}_{r-1} + \ldots + c_n\vec{x}_n$$

i.e. $\vec{x}_r$ is expressed as a linear combination of vectors.

$$\vec{x}_1, \vec{x}_2, \ldots \vec{x}_{r-1}, \vec{x}_{r+1}, \ldots \vec{x}_n$$

Hence $\vec{x}_r$ *with* $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_{r-1}, \vec{x}_{r+1} \ldots \vec{x}_n$ forms a linearly dependent set of vectors.

Note 2:

(i) If $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$ then $\vec{a}$ is expressed as a **Linear Combination** of vectors $\hat{i}, \hat{j}, \hat{k}$. Also, $\vec{a}$, $\hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors is a linearly dependent system.

(ii) $\hat{i}, \hat{j}, \hat{k}$ are **Linearly Independent** set of vectors. For

$$k_1\hat{i}+k_2\hat{j}+k_3\hat{k}=\vec{0} \Rightarrow k_1=0=k=k_3.$$

2. Show that the vectors $5\vec{a}+6\vec{b}+7\vec{c},\ 7\vec{a}-8\vec{b}+9\vec{c}$ and $3\vec{a}+20\vec{b}+5\vec{c}$ are coplanar (where $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors).

Solution.

Let $\vec{A}=5\vec{a}+6\vec{b}+7\vec{c}, \vec{B}=7\vec{a}-8\vec{b}+9\vec{c}$ and $\vec{C}=3\vec{a}+20\vec{b}+5\vec{c}$

$\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar $\Rightarrow x\vec{A}+y\vec{B}+z\vec{C}=\vec{0}$ must have a real solution for x, y, z other than (0,0,0).

Now $$x(5\vec{a}+6\vec{b}+7\vec{c})+y(7\vec{a}-8\vec{b}+9\vec{c})+z(3\vec{a}+20\vec{b}+5\vec{c})=\vec{0}$$

$$\Rightarrow (5x+7y+3z)\vec{a}+(6x-8y+20z)\vec{b}+(7x-9y+5z)\vec{c}=\vec{0}$$

$$5x+7y+3z=0$$
$$6x-8y+20z=0$$
$$7x+9y+5z=0 \qquad \text{(As } \vec{a}, \vec{b}, \vec{c} \text{ are non-coplanar vectors)}$$

Now, $$D=\begin{vmatrix} 5 & 7 & 3 \\ 6 & -8 & 20 \\ 7 & 9 & 5 \end{vmatrix}=0$$

So, the three linear simultaneous equation in x, y and z have a non-trivial solution.

Hence, $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar vectors.

6. COLLINEARITY AND COPLANARITY OF POINTS

(a) The necessary and sufficient condition for three points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to be collinear is that there exist scalars x, y, z, not all zero, such that, where $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$.

(b) The necessary and sufficient condition for four points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ to be coplanar is that then exist scalars x, y, z and u, not all zero, such that, where $x\vec{a}+y\vec{b}+z\vec{c}+u\vec{d}=\vec{0}$.

3. Let 'O' be the point of intersection of diagonals of a parallelogram $ABCD$. The points M, N, K and P are the mid points of OA, MB, NC and KD respectively. Show that N, O and P are collinear.

Solution.

Let $\quad O=(\vec{o}), A(\vec{a}), B(\vec{b})$

Now, $$M\equiv\frac{\vec{a}}{2},\ N\equiv\frac{\frac{\vec{a}}{2}+\vec{b}}{2}=\frac{\vec{a}+2\vec{b}}{4}$$

$$K\equiv\frac{\frac{\vec{a}+\vec{b}}{4}-\vec{a}}{2}=\frac{2\vec{b}-3\vec{a}}{8}$$

$$P\equiv\frac{-\vec{b}+\frac{2\vec{b}-3\vec{a}}{8}}{2}=\frac{-6\vec{b}-3\vec{a}}{16}$$

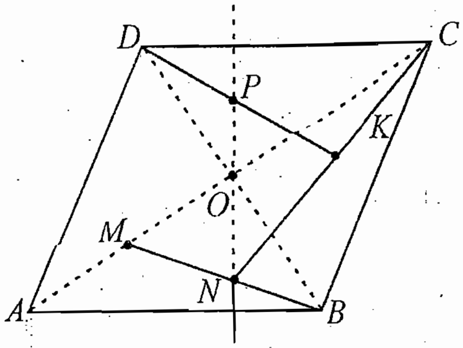


Figure 1.8

$$\Rightarrow \quad \overrightarrow{OP}=-\frac{3}{16}(2\vec{b}+\vec{a})$$

Also, $\overrightarrow{ON}=\frac{1}{4}(\vec{a}+2\vec{b})=-\frac{1}{6}(\overrightarrow{OP})$

Hence, points N, O and P are collinear.

7. SECTION FORMULA

Let A, B and C be three collinear points in space having position vectors $\vec{a}$, $\vec{b}$ and $\vec{r}$.

Let $\dfrac{AC}{CB} = \dfrac{n}{m}$

or, $mAC = nCB$

or, $m\overline{AC} = n\overline{CB}$ ... (i)

(As vectors are in same direction)

Now, $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC} \Rightarrow \overrightarrow{AC} = \vec{r} - \vec{a}$ ... (ii)

$\vec{r} + \overrightarrow{CB} = \vec{b} \Rightarrow \overrightarrow{CB} = \vec{b} - \vec{r}$ ... (iii)

Using (i), we get $\vec{r} = \dfrac{m\vec{a} + n\vec{b}}{m+n}$

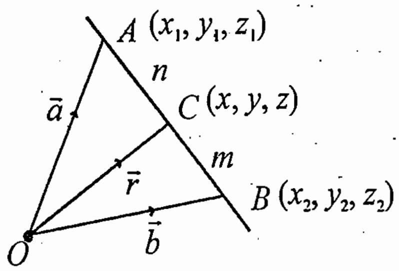


Figure 1.9

8. ORTHOGONAL SYSTEM OF UNIT VECTORS

Let OX, OY and OZ be three mutually perpendicular straight lines. Given any point $P(x, y, z)$ in space, we can construct the rectangular parallelepiped of which OP is a diagonal and $OA = x$, $OB = y$, $OC = z$.

Here A, B, C are $(x, 0, 0)$, $(0, y, 0)$ and $(0, 0, z)$ respectively and L, M, N are $(0, y, z)$, $(x, 0, z)$ and $(x, y, 0)$ respectively.

Let $\hat{i}, \hat{j}, \hat{k}$ denote unit vectors along OX, OY and OZ respectively.

We have $\vec{r} = \overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ as $\overrightarrow{OA} = x\hat{i}$, $\overrightarrow{OB} = y\hat{j}$ and $\overrightarrow{OC} = z\hat{k}$.

$$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$$

$$\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP}$$

So, $$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \quad (\overrightarrow{NP} = \overrightarrow{OC},\ \overrightarrow{AN} = \overrightarrow{OB})$$

$$r = |\vec{r}| = |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = \ell\hat{i} + m\hat{j} + n\hat{k}$$

$$\Rightarrow \vec{r} = \ell r\hat{i} + mr\hat{j} + nr\hat{k}$$

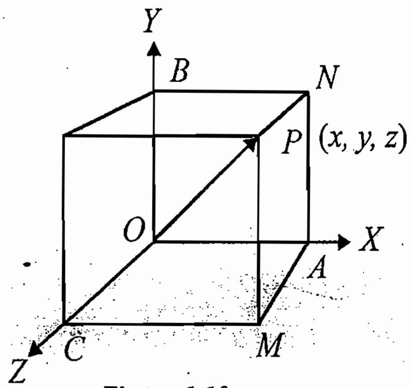


Figure 1.10

$$\ell = \cos\alpha = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$ (where α is the angle between OP and x-axis)

$$m = \cos\beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$ (where β is the angle between OP and y-axis)

$$n = \cos\gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$ (where γ is the angle between OP and z-axis)

ℓ, m, n are defined as the direction cosines of the line OP and x, y, z are defined as direction ratios of the line OP.

If $P \equiv (x_1, y_1, z_1)$ and $Q \equiv (x_2, y_2, z_2)$ then $\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Therefore, $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Hence, direction ratios of the line through P and Q are $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ and its direction cosines are $\dfrac{x_2 - x_1}{PQ}$, $\dfrac{y_2 - y_1}{PQ}$ and $\dfrac{z_2 - z_1}{PQ}$.

8.1 Some Properties of Direction Cosines and Ratios

(i) $\ell r, mr, nr$ are the projection of $\vec r$ on x, y and z-axis.

(ii) $\vec r = \ell\hat i + m\hat j + n\hat k$

(iii) $\ell^2 + m^2 + n^2 = 1$

(iv) If a, b and c are three real numbers such that $\dfrac{\ell}{a} = \dfrac{m}{b} = \dfrac{n}{c}$, then a, b, c are the direction ratios of the line whose direction cosines are ℓ, m and n.

9. SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT)

The scalar product, $\vec a\cdot\vec b$ of two non-zero vectors $\vec a$ and $\vec b$ is defined as $|\vec a||\vec b|\cos\theta$, where θ is angle between the two vectors, when drawn with same initial point.

Note that $0 \le \theta \le \pi$.

If at least one of $\vec a$ and $\vec b$ is a zero vector, then $\vec a\cdot\vec b$ is defined as zero.

9.1 Properties

(i) $\vec a\cdot\vec b = \vec b\cdot\vec a$ (scalar product is commutative)

(ii) $\vec a^2 = \vec a\cdot\vec a = |\vec a|^2 = a^2$

(iii) $(m\vec a)\cdot\vec b = m(\vec a\cdot\vec b) = \vec a(m\vec b)$ (where m is a scalar)

(iv) $\theta = \cos^{-1}\left(\dfrac{\vec a.\vec b}{|\vec a||\vec b|}\right)$

(v) $\vec a\cdot\vec b = 0 \Leftrightarrow$ Vectors $\vec a$ and $\vec b$ are perpendicular to each other. [$\vec a$, $\vec b$ are non-zero vectors].

(vi) $\hat i\cdot\hat j = \hat j\cdot\hat k = \hat k\cdot\hat i = 0$

(vii) $\vec a\cdot(\vec b+\vec c) = \vec a\cdot\vec b + \vec a\cdot\vec c$

(viii) $(\vec a+\vec b)\cdot(\vec a-\vec b) = |\vec a|^2 - |\vec b|^2 = a^2 - b^2$

(ix) Let $\vec a = a_1\hat i + a_2\hat j + a_3\hat k$, $\vec b = b_1\hat i + b_2\hat j + b_3\hat k$. Then $\vec a\cdot\vec b = (a_1\hat i + a_2\hat j + a_3\hat k)\cdot(b_1\hat i + b_2\hat j + b_3\hat k)$

(x) Maximum value of $\vec a.\vec b = |\vec a|\,|\vec b|$

(xi) Minimum value of $\vec a.\vec b = -|\vec a|\,|\vec b|$

(xii) Any vector $\vec a$ can be written as, $\vec a = (\vec a.\hat i)\hat i + (\vec a.\hat j)\hat j + (\vec a.\hat k)\hat k$

9.2 Algebraic Projection of a Vector Along Some other Vector

$$ON = OB\cos\theta = |\vec b|\frac{\vec a.\vec b}{|\vec a||\vec b|} = \hat a.\vec b$$

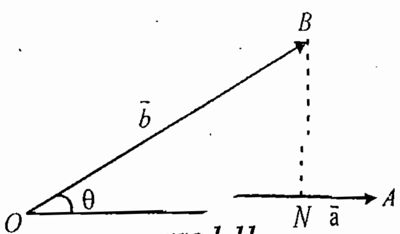


Figure 1.11

4. Prove that the angle in a semi-circle is a right angle.

Solution.

Let O be the centre and AB the bounding diameter of the semi-circle. Let P be any point on the circumference. With O as origin.

Let $\overrightarrow{OA} = \vec a$, $\overrightarrow{OB} = -\vec a$ and $\overrightarrow{OP} = \vec r$

Obviously, $OA = OB = OP$, each being equal to radius of the semi-circle.

$\overrightarrow{AP} = \vec{r} - \vec{a}$ and $\overrightarrow{BP} = \vec{r} - (-\vec{a}) = \vec{r} + \vec{a}$

$$\therefore \quad \overrightarrow{AP}\cdot\overrightarrow{BP} = (\vec{r}-\vec{a})\cdot(\vec{r}+\vec{a}) = r^2 - a^2$$
$$= OP^2 - OA^2 = 0$$

$\Rightarrow AP$ and BP are perpendicular to each other, *i.e.*, $\angle APB = 90^0$.

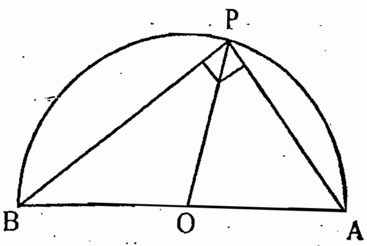


Figure 1.12

10. VECTOR (CROSS) PRODUCT

The vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$, whose magnitudes are a and b respectively, is the vector whose modulus is $ab\sin\theta$, where $\theta (0 \le \theta \le \pi)$ is the angle between vectors $\vec{a}$ and $\vec{b}$ as shown in figure 1.13.

Its direction is that of a vector $\hat{n}$ perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}, \hat{n}$ are in right-handed orientation.

By the right-handed orientation we mean that, if we turn the vector $\vec{a}$ into the vector $\vec{b}$ through the angle θ, then $\hat{n}$ points in the direction in which a right handed screw would move if turned in the same manner. Thus $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$. If at least one of $\vec{a}$ and $\vec{b}$ is a zero vector, then $\vec{a}\times\vec{b}$ is defined as the zero vector.

10.1 Properties

(i) $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

(ii) $(m\vec{a})\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times(m\vec{b})$ (where m is a scalar)

(iii) $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow$ vectors $\vec{a}$ and $\vec{b}$ are parallel. (provided $\vec{a}$ and $\vec{b}$ are non-zero vectors).

(iv) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

(v) $\hat{i}\times\hat{j} = \hat{k} = -(\hat{j}\times\hat{i}), \hat{j}\times\hat{k} = \hat{i} = -(\hat{k}\times\hat{j}), \hat{k}\times\hat{i} = \hat{j} = -(\hat{i}\times\hat{k})$

(vi) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

(vii) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}(a_2b_3 - a_3b_2) + \hat{j}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - a_2b_1)$$

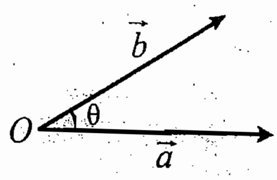


Figure 1.13

(viii) $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$.

(ix) Area of triangle $= \dfrac{1}{2}ap = \dfrac{1}{2}ab\sin\theta = \dfrac{1}{2}|\vec{a}\times\vec{b}|$

(x) Area of parallelogram $= ap = ab\sin\theta = |\vec{a}\times\vec{b}|$.

(xi) $\vec{a}\times\vec{b} \ne \vec{b}\times\vec{a}$ (not commutative)

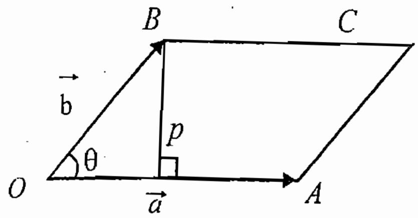


Figure 1.14

(xii) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

(xiii) A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is

$$\pm\frac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}.$$

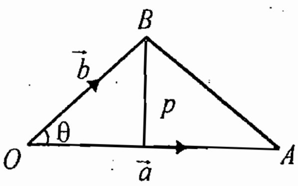


Figure 1.15

(xiv) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$.

(xv) Lagranges Identity : for any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$.

5. If $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $\vec{a}+\vec{b}+\vec{c}=0$, prove that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ and deduce the sine rule $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Solution.

Let $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$ represent the vectors $\vec{a}, \vec{b}, \vec{c}$ respectively.

Then, we have

$$\vec{a}+\vec{b}+\vec{c} = 0$$

$$\Rightarrow \quad \vec{c} = -(\vec{a}+\vec{b})$$

$$\Rightarrow \quad \vec{b}\times\vec{c} = \vec{b}\times(-\vec{a}-\vec{b})$$

$$= -\vec{b}\times\vec{a} = \vec{a}\times\vec{b}$$

Similarly, $\quad \vec{c}\times\vec{a} = \vec{a}\times\vec{b}$

Hence, $\quad \vec{b}\times\vec{c} = \vec{c}\times\vec{a} = \vec{a}\times\vec{b}$

$$\Rightarrow \quad bc\sin(\pi - A) = ca\sin(\pi - B) = ab\sin(\pi - C)$$

$$\Rightarrow \quad bc\sin A = ca\sin B = ab\sin C$$

$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

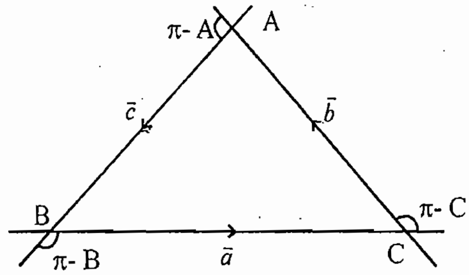


Figure 1.16

11. SCALAR TRIPLE PRODUCT (BOX PRODUCT)

The scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $(\vec{a}\times\vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$.

Let $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Then $\quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$$(\vec{a}\times\vec{b})\cdot\vec{c} = c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Therefore $\quad (\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a} = (\vec{c}\times\vec{a})\cdot\vec{b} = -(\vec{b}\times\vec{a})\cdot\vec{c} = -(\vec{c}\times\vec{b})\cdot\vec{a} = -(\vec{a}\times\vec{c})\cdot\vec{b}$

Note that $(\vec{a}\times\vec{b}).\vec{c} = (\vec{b}\times\vec{c}).\vec{a} = \vec{a}.(\vec{b}\times\vec{c})$, hence in scalar triple product dot and cross are interchangeable. Therefore we denote $(\vec{a}\times\vec{b}).\vec{c}$ by $[\vec{a}\ \vec{b}\ \vec{c}]$.

11.1 Properties

(i) $|(\vec{a}\times\vec{b}).\vec{c}|$ represents the volume of the parallelepiped, whose adjacent sides -- represented by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ in magnitude and direction. Therefore three vecto-- $a, b, \vec{c}$ are coplanar if and

only if $6[\vec{a}\ \vec{b}\ \vec{c}]=0$. i.e., $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$

(ii) Volume of the tetrahedron $=\frac{1}{6}|[\vec{a}\ \vec{b}\ \vec{c}]|$.

(iii) $[\vec{a}+\vec{b}\ \vec{c}\ \vec{d}]=[\vec{a}\ \vec{c}\ \vec{d}]+[\vec{b}\ \vec{c}\ \vec{d}]$

(iv) $[\vec{a}\ \vec{a}\ \vec{b}]=0$.

(v) In a scalar triple product the position of dot & cross can be interchanged *i.e.*

$$\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c} \quad or \quad [\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$$

(vi) $\vec{a}\cdot(\vec{b}\times\vec{c})=-\vec{a}\cdot(\vec{c}\times\vec{b})$ *i.e.* $[\vec{a}\ \vec{b}\ \vec{c}]=-[\vec{a}\ \vec{c}\ \vec{b}]$

(vii) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ & $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

then $[\vec{a}\vec{b}\vec{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

In general, if $\vec{a}=a_1\vec{l}+a_2\vec{m}+a_3\vec{n}$; $\vec{b}=b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$ & $\vec{c}=c_1\vec{l}+c_2\vec{m}+c_3\vec{n}$.

then $[\vec{a}\vec{b}\vec{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\vec{l}\ \vec{m}\ \vec{n}]$; where $\vec{\ell}$, $\vec{m}$ & $\vec{n}$ are non coplanar vectors.

(viii) If $\vec{a},\vec{b},\vec{c}$ are coplanar $\Leftrightarrow[\vec{a}\ \vec{b}\ \vec{c}]=0$.

(ix) Scalar product of three vectors, two of which are equal or parallel is 0 *i.e.* $[\vec{a}\vec{b}\vec{c}]=0$,

Note: If $\vec{a},\vec{b},\vec{c}$ are non-coplanar then $[\vec{a}\ \vec{b}\ \vec{c}]>0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}]<0$ for left handed system.

(x) $[\hat{i}\ \hat{j}\ \hat{k}]=1$.

(xi) $[K\vec{a}\ \vec{b}\ \vec{c}]=K[\vec{a}\ \vec{b}\ \vec{c}]$.

(xii) $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}]=[\vec{a}\ \vec{c}\ \vec{d}]+[\vec{b}\ \vec{c}\ \vec{d}]$.

(xiv) The volume of the tetrahedron $OABC$ with O as origin & the positive vectors of A, B and C being $\vec{a}$, $\vec{b}$ & $\vec{c}$ respectively is given by $V=\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

(xv) The position vector of the centroid of a tetrahedron if the positive vectors of its angular vertices are $\vec{a},\vec{b},\vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}[\vec{a}+\vec{b}+\vec{c}+\vec{d}]$.

Note that: This is also the point of concurrence of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

Remember that : $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]=0$ & $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]$.

6. Prove that the formula for the volume V of a tetrahedron in terms of the lengths a, b and c of three concurrent edges and their mutual inclinations ϕ, θ and ψ is given by

$$V^2=\begin{vmatrix} 1 & \cos\varphi & \cos\psi \\ \cos\varphi & 1 & \cos\theta \\ \cos\psi & \cos\theta & 1 \end{vmatrix}.$$

Solution.

Let $OABC$ be the tetrahedron with O as origin. Let $\vec{a},\vec{b},\vec{c}$ be the position vectors of A, B, C.

Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k},$

Then

$$V=\frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]=\frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$V^2=\frac{1}{36}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\times\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=\frac{1}{36}\begin{vmatrix} a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3 \\ a_1b_1+a_2b_2+a_3b_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3 \\ a_1c_1+a_2c_2+a_3c_3 & b_1c_1+b_2c_2+b_3c_3 & c_1^2+c_2^2+c_3^2 \end{vmatrix}$$

$$=\frac{1}{36}\begin{vmatrix} |a^2| & a.b & a.c \\ a.b & |b|^2 & b.c \\ a.c & b.c & |c|^2 \end{vmatrix}$$

$$=\frac{1}{36}\begin{vmatrix} a^2 & ab\cos\varphi & ca\cos\psi \\ ab\cos\varphi & b^2 & bc\cos\theta \\ ca\cos\psi & bc\cos\theta & c^2 \end{vmatrix}$$

$$=\frac{a^2b^2c^2}{36}\begin{vmatrix} 1 & \cos\varphi & \cos\psi \\ \cos\varphi & 1 & \cos\theta \\ \cos\psi & \cos\theta & 1 \end{vmatrix}$$

12. VECTOR TRIPLE PRODUCT

The vector triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as $\vec{a}\times(\vec{b}\times\vec{c})$. If at least one $\vec{a}$, $\vec{b}$ and $\vec{c}$ is a zero vector or $\vec{b}$ and $\vec{c}$ are collinear vectors or $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$, only then $\vec{a}\times(\vec{b}\times\vec{c})=\vec{0}$. In all other cases $\vec{a}\times(\vec{b}\times\vec{c})$ will be a non-zero vector in the plane of non-collinear vectors and perpendicular to the vector $\vec{a}$.

Thus we can take $\vec{a}\times(\vec{b}\times\vec{c})=\lambda\vec{b}+\mu\vec{c}$, for some scalars λ and μ. Since $\vec{a}\perp\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{a}.(\vec{a}\times(\vec{b}\times\vec{c}))=0\Rightarrow\lambda(\vec{a}.\vec{b})+\mu(\vec{a}.\vec{c})=0\Rightarrow\lambda(\vec{a}.\vec{c})\alpha,\ \mu=-(\vec{a}.\vec{b})\alpha$, for same scalar α.

Hence $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$, for any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfying the conditions given in the beginning. In particular if we take, $\vec{a}=\vec{b}=\hat{i},\ \vec{c}=\hat{j}$, then $\alpha=1$.

Hence $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

7. For any vector $\vec{a}$, prove that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$.

Solution.

$$[\hat{i}\times(\vec{a}\times\hat{i})]+[\hat{j}\times(\vec{a}\times\hat{j})]+[\hat{k}\times(\vec{a}\times\hat{k})]$$

$$=[(\hat{i}.\hat{i})\vec{a}-(\hat{i}.\vec{a})\hat{i}][(\hat{i}.\hat{j})\vec{a}-(\hat{j}.\vec{a})\hat{j}]+[(\hat{k}.\hat{k})\vec{a}-(\hat{k}.\vec{a})\hat{k}]$$

$$= \vec{a}-(\hat{i}.\vec{a})\hat{i}+\vec{a}-(\hat{j}.\vec{a})\hat{j}+\vec{a}-(\hat{k}.\vec{a})\hat{k} \qquad [\because \ \hat{i}.\hat{i}=\hat{j}\hat{j}=\hat{k}\hat{k}=1]$$

$$= 3\vec{a}-[(\hat{i}.\vec{a})\hat{i}+(\hat{j}.\vec{a})\hat{j}+(\hat{k}.\vec{a})\hat{k}]$$

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Then

$$\hat{i}.\vec{a} = \hat{i}.(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) = a_1\hat{i}^2+a_2(\hat{i}.\hat{j})+a_3(\hat{i}.\hat{k}) = a_1(1)+a_2(0)+a_3(0)=a$$

Similarly, $\hat{j}\cdot\hat{a} = a_2,\ \hat{k}\cdot\vec{a}=a_3$

$\therefore$ L.H.S. $= 3\vec{a}-(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) = 3\vec{a}-\vec{a}=2\vec{a}=$ R.H.S.

13. RECIPROCAL SYSTEM OF VECTORS

Let $\vec{a},\vec{b}$ and $\vec{c}$ be a system of three non-coplanar vectors. Then the system of vectors $\vec{a}',\vec{b}'$ and $\vec{c}'$ which satisfies $\vec{a}.\vec{a}'=\vec{b}.\vec{b}'=\vec{c}.\vec{c}'=1$ and $\vec{a}.\vec{b}'=\vec{a}.\vec{c}'=\vec{b}.\vec{a}'=\vec{b}.\vec{c}'=\vec{c}.\vec{a}'=\vec{c}.\vec{b}'=0$, is called the reciprocal system to the vectors $\vec{a},\vec{b},\vec{c}$. In term of $\vec{a},\vec{b},\vec{c}$ the vectors $\vec{a}',\vec{b}',\vec{c}'$ are given by

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{c}'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}.$$

13.1 Properties

(i) $\vec{a}\cdot\vec{b}'=\vec{a}\cdot\vec{c}'=\vec{b}\cdot\vec{a}'=\vec{b}\cdot\vec{c}'=\vec{c}\cdot\vec{a}'=\vec{c}\cdot\vec{b}'=0$

(ii) The scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ formed from three non-coplanar vectors $\vec{a},\vec{b},\vec{c}$ is the reciprocal of the scalar triple product formed from reciprocal system.

14. SOLVING OF VECTOR EQUATION

Solving a vector equation means determining an unknown vector (or a number of vectors satisfying the given conditions).

Generally, to solve vector equations, we express the unknown as the linear combination of three non-coplanar vectors as $\vec{r}=x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})$ as $\vec{a},\vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar and find x, y, z using given conditions.

Sometimes, we can directly solve the given condtions it would be more clear from some examples.

8. Solve the vector equation $\vec{r}\times\vec{b}=\vec{a}\times\vec{b}, \vec{r}.\vec{c}=0$ provided that $\vec{c}$ is not perpendicular to $\vec{b}$.

Solution.

We are given;

$$\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$$

$$\Rightarrow \quad (\vec{r}-\vec{a})\times\vec{b} = 0$$

Hence, $(\vec{r}-\vec{a})$ and $\vec{b}$ are parallel

$$\Rightarrow \quad \vec{r}-\vec{a} = t\vec{b} \qquad ...(i)$$

and we know $\quad \vec{r}.\vec{c}=0$

$\therefore$ Taking dot product of (i) by $\vec{c}$ we get

$$\vec{r}.\vec{c}-\vec{a}.\vec{c} = t(\vec{b}.\vec{c})$$

$$0-\vec{a}.\vec{c} = t(\vec{b}.\vec{c})$$

or

$$t=-\left(\frac{\vec{a}.\vec{c}}{\vec{b}.\vec{c}}\right) \qquad ...(ii)$$

$\therefore$ from (i) and (ii) solution of $\vec{r}$ is given by

$$\vec{r} = \vec{a} - \left(\frac{\vec{a}.\vec{c}}{\vec{b}.\vec{c}}\right)\vec{a}.$$

SOLVED EXAMPLES (OBJECTIVE)

1. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors, then $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2$ does not exceed

(a) 4 (b) 9 (c) 8 (d) 6

Ans. (b)

We have

$$0 \le |\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2|\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}|$$
$$= 3+2|\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}|$$

So, $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} \ge -3/2.$

Now, $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|+|\vec{c}-\vec{a}|^2$

$$= 2(|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2-(\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}) \le 2\,(3+3/2) = 9.$$

Hence (b) is the correct answer.

2. Let $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}$. If $\vec{c}$ is a vector such that $\vec{a}\cdot\vec{c} = |\vec{c}|, |\vec{c}-\vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$ is 30°, then $|(\vec{a}\times\vec{b})\times\vec{c}| =$

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) 2 (d) 3

Ans. (b)

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i}+2\hat{j}+\hat{k}$$

$\Rightarrow$ $|\vec{a}\times\vec{b}| = \sqrt{4+4+1} = 3$. Also $|\vec{c}-\vec{a}|^2 = 8$

So, $|\vec{c}|^2+|\vec{a}|^2-2\vec{a}\cdot\vec{c} = 8 \Rightarrow |\vec{c}|^2+|\vec{a}|^2-2|\vec{c}| = 8$

$|\vec{c}|^2+9-2|\vec{c}| = 8 \Rightarrow |\vec{c}|^2-2|\vec{c}|+1 = 0$

$(|\vec{c}|^2-1)^2 = 0 \Rightarrow |\vec{c}| = 1$

Now, $|(\vec{a}\times\vec{b})\times\vec{c}| = |\vec{a}\times\vec{b}|\times|\vec{c}|\sin 30° = 3.1.\frac{1}{2} = \frac{3}{2}$

Hence (b) is the correct answer.

3. Let $\vec{v} = 2\hat{i}+\hat{j}-\hat{k}$ and $\vec{w} = \hat{i}+3\hat{k}$. If $\hat{u}$ is a unit vector, then for the maximum value of the scalar triple product $[\vec{u}\ \vec{v}\ \vec{w}]$, $\vec{u} =$

(a) $\frac{1}{\sqrt{14}}(3\hat{i}+\hat{j}+2\hat{k})$ (b) $\frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}-\hat{k})$

(c) $\frac{1}{\sqrt{10}}(\hat{i}+3\hat{k})$ (d) $\frac{1}{\sqrt{59}}(3\hat{i}-7\hat{j}-\hat{k})$

Ans. (d)

$$\vec{v}\times\vec{w} = 3\hat{i}-7\hat{j}-\hat{k}$$

Now, $[\vec{u}\ \vec{v}\ \vec{w}] = \vec{u}\cdot(3\hat{i}-7\hat{j}-\hat{k})$

$= |\vec{u}||3\hat{i}-7\hat{j}-\hat{k}|\cos\theta$ (where θ is the angle between $\vec{u}$ and $\vec{v}\times\vec{w}$)

$= \sqrt{59}\cos\theta$

Thus $[\vec{u}\ \vec{v}\ \vec{w}]$ is maximum if $\cos\theta = 1$ *i.e.* $\theta = 0$ or $\vec{u} = \frac{1}{\sqrt{59}}(3\hat{i}-7\hat{j}-\hat{k})$

Hence (d) is the correct answer.

4. Let there be two points A and B on the curve $y = x^2$ in the plane OXY satisfying $OA.\,\hat{i} = 1$ and $\overrightarrow{OB}\cdot\hat{i} = -2$ then the length of the vector $2\overrightarrow{OA} - 3\overrightarrow{OB}$ is

(a) $\sqrt{14}$ (b) $2\sqrt{51}$ (c) $3\sqrt{41}$ (d) $2\sqrt{41}$

Ans. (d)

Let $\overrightarrow{OA} = x_1\hat{i} + y_1\hat{j}$ and $\overrightarrow{OB} = x_2\hat{i} + y_2\hat{j}$. Since $1 = \overrightarrow{OA}\cdot\hat{i} = x_1$ and $-2 = \overrightarrow{OB}\cdot\hat{i} = x_2$.

Moreover, $y_1 = x_1^2$ and $y_2 = x_2^2 = 4$, so $\overrightarrow{OA} = \hat{i} + \hat{j}$ and $\overrightarrow{OB} = -2\hat{i} + 4\hat{j}$.

Hence, $|2\overrightarrow{OA} - 3\overrightarrow{OB}| = |8\hat{i} - 10\hat{j}| = \sqrt{164} = 2\sqrt{41}$

Hence (d) is the correct answer.

5. If A, B, C and D are four points in space satisfying $\overrightarrow{AB}\cdot\overrightarrow{CD} = k[|\overrightarrow{AD}|^2 + |\overrightarrow{BC}|^2 - |\overrightarrow{AC}|^2 - |\overrightarrow{BD}|^2]$ then the value of k is

(a) 2 (b) 1/3 (c) 1/2 (d) 1

Ans. (c)

Let A be the origin of reference and let the position vectors of B, C, D be $\vec{b}, \vec{c}, \vec{d}$.

So, $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{CD} = \vec{d} - \vec{c}$, $\overrightarrow{AD} = \vec{d}$, $\overrightarrow{BC} = \vec{c} - \vec{b}$, $\overrightarrow{AC} = \vec{c}$ and $\overrightarrow{BD} = \vec{d} - \vec{b}$.

The L.H.S. is equal to $\vec{b}\cdot(\vec{d}\cdot\vec{c})$.

The R.H.S is

$$k[|\vec{d}|^2 + |\vec{c} - \vec{b}|^2 - |\vec{c}|^2 - |\vec{d} - \vec{b}|^2] = k[\vec{d}\cdot\vec{d} + \vec{c}\cdot\vec{c} + \vec{b}\cdot\vec{b} - 2\vec{c}\cdot\vec{b} - \vec{c}\cdot\vec{c} - \vec{d}\cdot\vec{d} - \vec{b}\cdot\vec{b} + 2d.\vec{b}]$$
$$= 2k[\vec{b}\cdot(\vec{d} - \vec{c})].$$

Hence (c) is the correct answer.

6. Three non-coplanar vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are drawn from a common initial point. The angle between the plane passing through the terminal points of these vectors and the vector $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) none of these.

Ans. (b)

Let the terminal points be A, B, C and the common initial point be the origin of reference so that $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = \vec{c} - \vec{a}$. The vector $\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicular to the plane ABC.

$\overrightarrow{AB}\times\overrightarrow{AC} = (\vec{b} - \vec{a})\times(\vec{c} - \vec{a}) = \vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}$.

Hence the angle between the plane and the given vector is $\pi/2$.

Hence (b) is the correct answer.

7. If $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 3\,|\vec{b}| = 5$ and $|\vec{c}| = -7$, then the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{3}$

Ans. (c)

$$\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow (\vec{a} + \vec{b})\cdot(\vec{a} + \vec{b}) = |\vec{c}|^2$$

Thus $|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = |\vec{c}|^2$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

therefore $\cos\theta = \dfrac{49 - 9 - 25}{2.3.5} = \dfrac{1}{2}$

$$\Rightarrow \qquad \theta = \frac{\pi}{3}$$

Hence (c) is correct answer.

8. Given that $(\vec{x}-\hat{a})\cdot(\vec{x}+\hat{a})=8$ and $\vec{x}\cdot\hat{a}=2$, then the angle between $(\vec{x}-\hat{a})$ & $(\vec{x}+\hat{a})$ is :

(a) $\cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$ (b) $\cos^{-1}\left(\frac{3}{\sqrt{21}}\right)$ (c) $\cos^{-1}\left(\frac{5}{\sqrt{21}}\right)$ (d) none of these

Ans. (d)

$$(\vec{x}-\hat{a})\cdot(\vec{x}+\hat{a})=8 \quad\Rightarrow\quad x=3$$

To determine $(\vec{x}-\hat{a})$ we have

$$(\vec{x}-\hat{a})\cdot(\vec{x}-\hat{a})=9+1-4=6$$

so that $|\vec{x}-\hat{a}|=\sqrt{6}$ and similarly $|\vec{x}+\hat{a}|=\sqrt{14}$

Then $(\vec{x}-\hat{a})\cdot(\vec{x}+\hat{a})=\sqrt{14}\,\sqrt{6}\cos\theta$

$$8=\sqrt{14}\,\sqrt{6}\cos\theta$$

$$\cos\theta=\frac{4}{\sqrt{21}}$$

9. The vector $(\vec{a}+3\vec{b})$ is perpendicular to $(7\vec{a}-5\vec{b})$ and $(\vec{a}-4\vec{b})$ is perpendicular to $(7\vec{a}-2\vec{b})$. The angle between $\vec{a}$ & $\vec{b}$ is :

(a) 30° (b) 45° (c) 60° (d) none of these

Ans. (c)

Given $(\vec{a}+3\vec{b})\cdot(7\vec{a}+5\vec{b})=0$

$\Rightarrow\ 7a^2-15b^2+16\,\vec{a}\cdot\vec{b}=0$ ...(1)

Also, $(\vec{a}-4\vec{b})\cdot(7\vec{a}-2\vec{b})=0$

$\Rightarrow\ 7a^2+8b^2-30\,\vec{a}\cdot\vec{b}=0$ ...(2)

Subtracting, $-23b^2+46\,\vec{a}\cdot\vec{b}=0 \quad\Rightarrow\quad \vec{a}\cdot\vec{b}=\frac{b^2}{2}$

Putting this in (1),

$$7a^2-7b^2=0 \quad\Rightarrow\quad |\vec{a}|=|\vec{b}|$$

Thus $\vec{a}\cdot\vec{b}=ab\cos\theta$

$\Rightarrow\ \frac{b^2}{2}=b^2\cos\theta \Rightarrow \cos\theta=\frac{1}{2}$

or $\theta=60°$.

10. X-component of $\vec{a}$ is twice its Y-component. If the magnitude of the vector is $5\sqrt{2}$ and it makes an angle of 135° with z-axis then the vector is :

(a) $2\sqrt{3}, \sqrt{3}, -3$ (b) $2\sqrt{6}, \sqrt{6}, -6$ (c) $2\sqrt{5}, \sqrt{5}, -5$ (d) none of these

Ans. (c)

Let $\vec{a}=2x\hat{i}+x\hat{j}+z\hat{k}$

$$\sqrt{5x^2+z^2}=5\sqrt{2}$$

Also, $\cos 135°=\frac{z}{\sqrt{5x^2+z^2}}=\frac{z}{5\sqrt{2}}=-\frac{1}{\sqrt{2}}$

$\Rightarrow\ z=-5$

then $x=\sqrt{5}$.

The required vector $\vec{a}=2\sqrt{5}\,\hat{i}+\sqrt{5}\,\hat{j}-5\hat{k}$

11. If $|\vec{a}\cdot\vec{b}|=|\vec{a}\times\vec{b}|$, then the angle between $\vec{a}$ & $\vec{b}$ is :

(a) 0° (b) 180° (c) 135° (d) 45°

Ans. (c), (d)

We have $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$

$\Rightarrow$ $|\vec{a}|\,|\vec{b}|\cos\theta = |\vec{a}|\,|\vec{b}|\sin\theta$

$\Rightarrow$ $\cos\theta = \sin\theta \Rightarrow \theta = 45°, 135°$

12. If the non zero vectors $\vec{a}$ & $\vec{b}$ are perpendicular to each other then the solution of the equation, $\vec{r}\times\vec{a} = \vec{b}$ is:

(a) $\vec{r} = x\vec{a} + \dfrac{1}{\vec{a}\,.\,\vec{a}}(\vec{a}\times\vec{b})$ (b) $\vec{r} = x\vec{b} - \dfrac{1}{\vec{b}\,.\,\vec{b}}(\vec{a}\times\vec{b})$

(c) $\vec{r} = x(\vec{a}\times\vec{b})$ (d) none of these

Ans. (a)

Since $\vec{a}$, $\vec{b}$ and $\vec{r}\times\vec{a}$ are non coplanar

$\therefore$ $$\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b}) \quad ...(1)$$

For some scalars x, y, z

Now $\vec{b} = \vec{r}\times\vec{a}$

$$\therefore \quad \vec{b} = \{x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})\}\times\vec{a}$$

$$= x(\vec{a}\times\vec{a}) + y(\vec{b}\times\vec{a}) + z\{(\vec{a}\times\vec{b})\times\vec{a}\}$$

$$= 0 + y(\vec{b}\times\vec{a}) + z\{(\vec{a}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}\}$$

$$\therefore \quad \vec{b} = y(\vec{b}\times\vec{a}) + z(\vec{a}\cdot\vec{a})\vec{b} \quad \{\because\ \vec{a}\cdot\vec{b} = 0\}$$

Comparing the coefficients, we get

$$y = 0 \text{ and } z = \frac{1}{(\vec{a}\,.\,\vec{a})}$$

Putting the values of y and z in (1), we get

$$\vec{r} = x\vec{a} + \frac{1}{(\vec{a}\cdot\vec{a})}(\vec{a}\times\vec{b})$$

13. If $\vec{a}$, $\vec{b}$ & $\vec{c}$ are non coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \dfrac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ & $\vec{b}$ is:

(a) $\dfrac{3\pi}{4}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{2}$ (d) π

Ans. (a)

We have $$\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$$

$$\Rightarrow (\vec{a}\,.\,\vec{c})\vec{b} - (\vec{a}\,.\,\vec{b})\vec{c} = \frac{\vec{b}}{\sqrt{2}} + \frac{\vec{c}}{\sqrt{2}}$$

$$\Rightarrow \left(\vec{a}\,.\,\vec{c} - \frac{1}{\sqrt{2}}\right)\vec{b} - \left(\vec{a}\,.\,\vec{b} + \frac{1}{\sqrt{2}}\right)\vec{c} = 0$$

then $\vec{a}\,.\,\vec{c} - \dfrac{1}{\sqrt{2}} = 0$ and $\vec{a}\,.\,\vec{b} + \dfrac{1}{\sqrt{2}} = 0$

$$\therefore \quad \vec{a}\,.\,\vec{b} = -\frac{1}{\sqrt{2}} \qquad (\because\ \vec{a}, \vec{b}, \vec{c} \text{ are non coplanar})$$

Let the angle between $\vec{a}$ & $\vec{b}$ be θ then

$$\vec{a}\cdot\vec{b} = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad |\vec{a}|\,|\vec{b}|\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad 1.1\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = \frac{3\pi}{4}$$

14. Let $\vec{a}$ & $\vec{b}$ are two vectors making angle θ with each other, then unit vectors along bisector of $\vec{a}$ & $\vec{b}$ is :

(a) $\pm\dfrac{\hat{a}+\hat{b}}{2}$ (b) $\pm\dfrac{\hat{a}+\hat{b}}{2\cos\theta}$ (c) $\pm\dfrac{\hat{a}+\hat{b}}{2\cos\theta/2}$ (d) $\dfrac{(\hat{a}+\hat{b})}{|\hat{a}+\hat{b}|}$

Solution.

Now in ΔABC

$$\frac{BD}{DC} = \frac{a}{b}$$

$\therefore\ BD = ak,\ DC = bk$

$$\therefore \quad \overline{BC} = (\vec{a}+\vec{b})k$$

$$(BC)^2 = (AB)^2 + (AC)^2 - 2AB.AC\cos\theta$$

$$\Rightarrow \quad (a+b)^2k^2 = a^2 + b^2 - 2ab\cos\theta$$

$$\therefore \quad k^2 = \frac{a^2+b^2-2ab\cos\theta}{(a+b)^2} \qquad .....(1)$$

In ΔADC and ΔABD

$$\cos\frac{\theta}{2} = \frac{b^2+(AD)^2-b^2k^2}{2b\,AD} = \frac{a^2+(AD)^2-a^2k^2}{2a\,AD}$$

$$\Rightarrow \quad (AD)^2 = ab\,(1-k^2)$$

$$= ab\left\{1-\frac{a^2+b^2-2ab\cos\theta}{(a+b)^2}\right\} \qquad \text{[from (1)]}$$

$$= \frac{4a^2b^2\cos^2\frac{\theta}{2}}{(a+b)^2}$$

$$\therefore \quad AD = \frac{2ab\cos\frac{\theta}{2}}{(a+b)}$$

$$\therefore \quad \overline{AD} = \pm\frac{(\vec{a}b+\vec{b}a)}{(a+b)}$$

$$= \pm\frac{ab}{(a+b)}\left(\frac{\vec{a}}{a}+\frac{\vec{b}}{b}\right)$$

$$= \pm\frac{ab}{(a+b)}(\hat{a}+\hat{b})$$

$$\therefore \quad \hat{AD} = \frac{\overrightarrow{AD}}{AD} = \pm\frac{(\hat{a}+\hat{b})}{2\cos\frac{\theta}{2}}$$

SOLVED EXAMPLES (SUBJECTIVE)

1. Find unit vector of $\hat{i}-2\hat{j}+3\hat{k}$

Solution. $\quad \vec{a} = \hat{i}-2\hat{j}+3\hat{k}$

If $\quad \vec{a} = a_x\hat{i}+a_y\hat{j}+a_z\hat{k}$

Then $\quad |\vec{a}| = \sqrt{a_x^2+a_y^2+a_z^2}$

$\therefore \quad |\vec{a}| = \sqrt{14}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}\,\hat{i} - \frac{2}{14}\,\hat{j} + \frac{3}{\sqrt{14}}\,\hat{k}$$

2. Find values of x & y for which the vectors

$\vec{a} = (x+2)\hat{i} - (x-y)\hat{j} + \hat{k}$

$\vec{b} = (x-1)\hat{i} + (2x+y)\hat{j} + 2\hat{k}$ are parallel.

Solution.

$\vec{a}$ and $\vec{b}$ are parallel if

$$\frac{x+2}{x-1} = \frac{y-x}{2x+y} = \frac{1}{2}$$

$x = -5,\ y = -20$

3. If $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

Solution.

Let $ABCD$ be a parallelogram such that $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$.

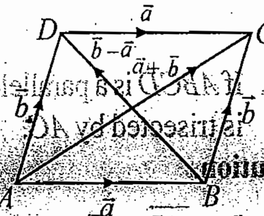


Figure 1.17

Then, $\quad \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\Rightarrow \quad \overrightarrow{AC} = \vec{a}+\vec{b} = 3\hat{i}+6\hat{j}-2\hat{k}$

and $\quad \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

$\Rightarrow \quad \overrightarrow{AD} + \overrightarrow{AD} = \overrightarrow{AB}$

$\Rightarrow \quad \overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB} = \vec{b}-\vec{a}$

Now, $\quad \overrightarrow{AC} = 3\hat{i}+6\hat{j}-2\hat{k}$

$\Rightarrow \quad |\overrightarrow{AC}| = \sqrt{9+36+4} = 7$

and, $\quad \overrightarrow{BD} = \hat{i}+2\hat{j}-8\hat{k}$

$\Rightarrow \quad |\overrightarrow{BD}| = \sqrt{1+4+64} = \sqrt{69}$

$\therefore$ Unit vector along $\overrightarrow{AC} = \dfrac{\overrightarrow{AC}}{|\overrightarrow{AC}|} = \dfrac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$

and, unit vector along $\overrightarrow{BD} = \dfrac{1}{\sqrt{69}}(\hat{i}+2\hat{j}-8\hat{k})$

4. $ABCDE$ is a pentagon. Prove that the resultant of the forces $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.

Solution.

Let $\vec{R}$ be the resultant force

$\therefore \quad \vec{R} = \overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC}$

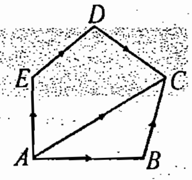


Figure 1.18

$$\therefore \qquad \vec{R} = (\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}) + \overrightarrow{AC}$$
$$= \overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{AC}$$
$$= 3\,\overrightarrow{AC}. \text{ Hence proved.}$$

5. *ABCD* is a parallelogram. If *L*, *M* be the middle point of *BC* and *CD*, express $\overline{AL}$ and $\overline{AM}$ in terms of $\overrightarrow{AB}$ and $\overline{AD}$, also show that $\overline{AL} + \overline{AM} = \frac{3}{2}\,\overline{AC}$.

Solution.

Let the position vectors of points *B* and *D* be respectively $\vec{b}$ and $\vec{a}$ referred to *A* as origin of reference.

Then $\qquad \overline{AC} = \overline{AD} + \overline{DC} = \overline{AD} + \overline{AB} \qquad [\because \overline{DC} = \overline{AB}]$

$\qquad = \vec{d} + \vec{b} \qquad \because \overline{AB} = \vec{b},\ \overline{AD} = \vec{d}$

i.e. position vector of *C* referred to *A* is $\vec{d} + \vec{b}$

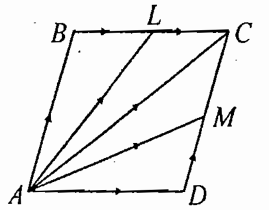


Figure 1.19

$$\therefore \qquad \overline{AL} = \text{p.v. of } L, \text{ the mid point of } \overline{BC}.$$
$$= \frac{1}{2}\,[\text{p.v. of } D + \text{p.v. of } C]$$
$$= \frac{1}{2}(\vec{b} + \vec{d} + \vec{b}) = \overline{AB} + \frac{1}{2}\,\overline{AD}$$
$$\overline{AM} = \frac{1}{2}\left[\vec{d} + \vec{d} + \vec{b}\right] = \overline{AD} + \frac{1}{2}\,\overline{AB}$$
$$\therefore \qquad \overline{AL} + \overline{AM} = \vec{b} + \frac{1}{2}\vec{d} + \vec{d} + \frac{1}{2}\vec{b}$$
$$= \frac{3}{2}\vec{b} + \frac{3}{2}\vec{d}$$
$$= \frac{3}{2}(\vec{b} + \vec{d}) = \frac{3}{2}\,\overline{AC}.$$

6. If *ABCD* is a parallelogram and *E* is the mid point of *AB*, show by vector method that *DE* trisects and is trisected by *AC*.

Solution.

Let $\overline{AB} = \vec{a}$ and $\overline{AD} = \vec{b}$

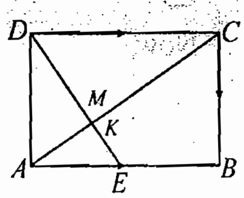


Figure 1.20

Then $\qquad \overline{BC} = \overline{AD} = \vec{b}$

and $\qquad \overline{AC} = \overline{AB} + \overline{AD} = \vec{a} + \vec{b}$

Also let *K* be a point on *AC*, such that $AK : AC = 1 : 3$

or, $\qquad AK = \frac{1}{3}\,AC$

$$\Rightarrow \qquad \overline{AK} = \frac{1}{3}(\vec{a} + \vec{b}) \qquad \text{.....(i)}$$

Again *E* being the mid point of *AB*, we have

$$\overline{AE} = \frac{1}{2}\vec{a}$$

Let *M* be the point on *DE* such that $DM : ME = 2 : 1$

$$\therefore \qquad \overline{AM} = \frac{\overline{AD} + 2\overline{AE}}{1+2} = \frac{\vec{b} + \vec{a}}{3} \qquad \text{.....(ii)}$$

From (i) and (ii) we find that

$$\overrightarrow{AK} = \frac{1}{3}(\vec{a} + \vec{b}) = \overrightarrow{AM},$$

and so we conclude that K and M coincide. i.e. DE trisect AC and is trisected by AC. Hence proved.

7. Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are

(i) perpendicular (ii) parallel

Solution.

(i) $\vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0 \Rightarrow (3\hat{i}+2\hat{j}+9\hat{k}).(\hat{i}+p\hat{j}+3\hat{k}) = 0$

$\Rightarrow 3 + 2p + 27 = 0 \Rightarrow p = -15$

(ii) We know that the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are parallel iff

$\vec{a} = \lambda\vec{b} \Leftrightarrow (a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) = \lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \Leftrightarrow a_1 = \lambda b_1, a_2 = \lambda b_2, a_3 = \lambda b_3$

$$\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} (= \lambda)$$

So, vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \Rightarrow 3 = \frac{2}{p} \Rightarrow p = 2/3$$

8. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution.

We have, $\vec{a}+\vec{b}+\vec{c} = \vec{0}$

$\Rightarrow \vec{a}+\vec{b} = -\vec{c}$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = (-\vec{c}).(-\vec{c})$

$\Rightarrow |\vec{a}+\vec{b}|^2 = |\vec{c}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{c}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = |\vec{c}|^2$

$\Rightarrow 9 + 25 + 2(3)(5)\cos\theta = 49$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}.$

9. Find the values of x for which the angle between the vectors $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse.

Solution.

The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Now, θ is obtuse $\Rightarrow \cos\theta < 0$

$\Rightarrow \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} < 0$

$\Rightarrow \vec{a}.\vec{b} < 0 \quad [\because |\vec{a}|, |\vec{b}| > 0]$

$\Rightarrow 14x^2 - 8x + x < 0$

$\Rightarrow \quad 17x(2x-1) < 0$

$\Rightarrow \quad x(2x-1) < 0$

$\Rightarrow \quad 0 < x < \dfrac{1}{2}$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$

10. D is the mid point of the side BC of a triangle ABC, show that $AB^2 + AC^2 = 2(AD^2 + BD^2)$.

Solution.

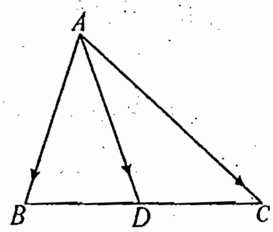


Figure 1.21

We have

$$\overline{AB} = \overline{AD} + \overline{DB}$$

$$\Rightarrow \quad AB^2 = (\overline{AD} + \overline{DB})^2$$

$$= AD^2 + DB + 2\overline{AD}.\overline{DB} \qquad \text{...(i)}$$

Also we have

$$\overline{AC} = \overline{AD} + \overline{DC}$$

$$\Rightarrow \quad AC^2 = (\overline{AD} + \overline{DC})^2$$

$$= AD^2 + DC^2 + 2\overline{AD}\ .\ \overline{DC} \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$AB^2 + AC^2 = 2AD^2 + 2BD^2 + 2\,\overline{AD}\ .\ (\overline{DB} + \overline{DC})$$

$$= 2(DA^2 + DB^2), \text{ for } \overline{DB} + \overline{DC} = 0$$

11. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, then find

(i) Component of $\vec{b}$ along $\vec{a}$.

(ii) Component of $\vec{b}$ perpendicular to along $\vec{a}$.

Solution.

(i) Component of $\vec{b}$ along $\vec{a}$ is

$$\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

Here $\vec{a}.\vec{b} = 2 - 1 + 3 = 4$

$|\vec{a}|^2 = 3$

Hence $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \dfrac{4}{3}\vec{a} = \dfrac{4}{3}(\hat{i} + \hat{j} + \hat{k})$.

(ii) Component of $\vec{b}$ perpendicular to along $\vec{a}$ is $\vec{b} - \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \dfrac{1}{3}(2\hat{i} - 7\hat{j} + 5\hat{k})$

12. Find a vector of magnitude 9, which is perpendicular to both the vectors $4\hat{i} + \hat{j} + 3\hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$.

Solution.

Let $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$. Then,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = (2-3)\hat{i} - (-8+6)\hat{j} + (4-2)\hat{k} = -\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\Rightarrow \quad |\vec{a}\times\vec{b}| = \sqrt{(-1)^2+2^2+2^2} = 3$$

$$\therefore \quad \text{Required vector} = 9\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right) = \frac{9}{3}(-\hat{i}+2\hat{j}+2\hat{k}) = -3\hat{i}+6\hat{j}+6\hat{k}$$

13. For any three vectors $\vec{a}, \vec{b}, \vec{c}$. Show that $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=\vec{0}$.

Solution.

We have, $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$

$$= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b} \quad \text{[Using distributive law]}$$

$$= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c} \quad [\because \vec{a}\times\vec{b}=-\vec{b}\times\vec{a} \text{ etc}]$$

14. For any vector $\vec{a}$, prove that $|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=2|\vec{a}|^2$

Solution.

Let $\quad \vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}.$

Then $\quad \vec{a}\times\hat{i} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{i}$

$$= a_1(\hat{i}\times\hat{i})+a_2(\hat{j}\times\hat{i})+a_3(\hat{k}\times\hat{i}) = -a_2\hat{k}+a_3\hat{i}$$

$$\Rightarrow \quad |\vec{a}\times\hat{i}|^2 = a_2^2+a_3^2$$

$$\vec{a}\times\hat{j} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{j} = a_1\hat{k}-a_3\hat{i}$$

$$\Rightarrow \quad |\vec{a}\times\hat{j}|^2 = a_1^2+a_3^2$$

$$\Rightarrow \quad |\vec{a}\times\hat{k}|^2 = a_1^2+a_2^2$$

$$\therefore \quad |\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2$$

$$= a_2^2+a_3^3+a_1^2+a_3^2+a_1^2+a_2^2$$

$$= 2(a_1^2+a_2^2+a_3^2) = 2|\vec{a}|^2$$

15. Let $\overline{OA}=\vec{a}$, $\overline{OB}=10\vec{a}+2\vec{b}$ and $\overline{OC}=\vec{b}$ where O is origin. Let p denote the area of the quadrilateral $OABC$ and q denote the area of the parallelogram with OA and OC as adjacent sides. Prove that $p=6q$.

Solution.

We have,

p = Area of the quadrilateral $OABC$

$$= \frac{1}{2}|\overline{OB}\times\overline{AC}|$$

$$= \frac{1}{2}|\overline{OB}\times(\overline{OC}-\overline{OA})|$$

$$= \frac{1}{2}|(10\vec{a}+2\vec{b})\times(\vec{b}-\vec{a})|$$

$$= \frac{1}{2}|10(\vec{a}\times\vec{b})-10(\vec{a}\times\vec{a})+2(\vec{b}\times\vec{b})-2(\vec{b}\times\vec{a})|$$

$$= \frac{1}{2}|10(\vec{a}\times\vec{b})-0+0+2(\vec{a}\times\vec{b})| \quad \text{....(i)}$$

and, q = Area of the parallelogram with OA and OC as adjacent sides

$$= |\overline{OA}\times\overline{OC}| = |\vec{a}\times\vec{b}| \quad \text{....(ii)}$$

From (i) and (ii), we get $p=6q$

16. Find the volume of a parallelopiped whose sides are given by $-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$.

Solution.

Let $\vec{a}=-3\hat{i}+7\hat{j}+5\hat{k}$, $\vec{b}=-5\hat{i}+7\hat{j}+3\hat{k}$ and $\vec{c}=7\hat{i}-5\hat{j}-3\hat{k}$.

We know that the volume of a parallelopiped whose three adjacent edges are $\vec{a},\vec{b},\vec{c}$ is $\left|[\vec{a},\vec{b},\vec{c}]\right|$.

Now, $$[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}-3 & 7 & 5\\ -5 & 7 & -3\\ 7 & -5 & -3\end{vmatrix}=-3(-21-15)-7(15+21)+5(25-49)$$

$$=108-252-120=-264$$

So, required volume of the parallelopiped $=\left|[\vec{a},\vec{b},\vec{c}]\right|=|-264|=264$ cubic units

17. Simplify $[\vec{a}-\vec{b}\ \vec{b}-\vec{c}\ \vec{c}-\vec{a}]$

Solution.

We have :

$[\vec{a}-\vec{b}\ \vec{b}-\vec{c}\ \vec{c}-\vec{a}]=\{(\vec{a}-\vec{b})\times(\vec{b}-\vec{c})\}\cdot(\vec{c}-\vec{a})$ [by def.]

$=(\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{b}+\vec{b}\times\vec{c})\cdot(\vec{c}-\vec{a})$ [by dist. law]

$=(\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+\vec{b}\times\vec{c})\cdot(\vec{c}-\vec{a})$ [since $\vec{b}\times\vec{b}=0$]

$=(\vec{a}\times\vec{b}).\vec{c}-(\vec{a}\times\vec{b}).\vec{a}+(\vec{c}\times\vec{a}).\vec{c}-(\vec{c}\times\vec{a}).\vec{a}+(\vec{b}\times\vec{c}).\vec{c}-(\vec{b}\times\vec{c}).\vec{a}$ [by dist. law]

$=[\vec{a}\,\vec{b}\,\vec{c}]-[\vec{a}\,\vec{b}\,\vec{a}]+[\vec{c}\,\vec{a}\,\vec{c}]-[\vec{c}\,\vec{a}\,\vec{a}]+[\vec{b}\,\vec{c}\,\vec{c}]-[\vec{b}\,\vec{c}\,\vec{a}]$

$=[\vec{a}\,\vec{b}\,\vec{c}]-[\vec{b}\,\vec{c}\,\vec{a}]$

[$\because$ scalar triple product when any two vectors are equal is zero]

$=[\vec{a}\,\vec{b}\,\vec{c}]-[\vec{a}\,\vec{b}\,\vec{c}]=0$

[Since $[\vec{b}\,\vec{c}\,\vec{a}]=[\vec{a}\,\vec{b}\,\vec{c}]$]

18. Find the volume of the tetrahedron whose four vertices have position vectors $\vec{a}\ \vec{b}\ \vec{c}$ and $\vec{d}$.

Solution.

Let four vertices be A, B, C, D with p. v. $\vec{a}\ \vec{b}\ \vec{c}$ and $\vec{d}$, respectively.

$\therefore$ $\overrightarrow{DA}=(\vec{a}-\vec{d})$

$\overrightarrow{DB}=(\vec{b}-\vec{d})$

$\overrightarrow{DC}=(\vec{c}-\vec{d})$

Hence volume $=\dfrac{1}{6}[\vec{a}-\vec{d}\ \vec{b}-\vec{d}\ \vec{c}-\vec{d}]$

$=\dfrac{1}{6}(\vec{a}-\vec{d}).[(\vec{b}-\vec{d})\times(\vec{c}-\vec{d})]$

$=\dfrac{1}{6}(\vec{a}-\vec{d}).[\vec{b}\times\vec{c}-\vec{b}\times\vec{d}+\vec{c}\times\vec{d}]$

$=\dfrac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{d}]+[\vec{a}\,\vec{c}\,\vec{d}]-[\vec{d}\,\vec{b}\,\vec{c}]\}$

19. Show that the vectors $\vec{a}=-2\vec{i}+4\vec{j}-2\vec{k}$, $\vec{b}=4\vec{i}-2\vec{j}-2\vec{k}$ and $\vec{c}=-2\vec{i}-2\vec{j}+4\vec{k}$ are coplanar.

Solution.

The vectors are coplanar since $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -2 & 4 & -2 \\ 4 & -2 & -2 \\ -2 & -2 & 4 \end{vmatrix} = 0$

20. For any vector $\vec{a}$, prove that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k}) = 2\vec{a}$

Solution.

Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$.

Then, $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\hat{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$

$$= \{(\hat{i}\,.\,\hat{i})\vec{a}-(\hat{i}\,.\,\vec{a})\hat{i}\} + \{(\hat{j}\,.\,\hat{j})\vec{a}-(\hat{j}\,.\,\vec{a})\hat{j}\} + \{(\hat{k}\,.\,\hat{k})\vec{a}-(\hat{k}\,.\,\vec{a})\hat{k}\}$$
$$= \{(\vec{a}-(\hat{i}\,.\,\vec{a})\,\hat{i}\} + \{\vec{a}-(\hat{j}\,.\,\vec{a})\hat{j}\} + \{\vec{a}-(\hat{k}\,.\,\vec{a})\hat{k}\}$$
$$= 3\vec{a}-\{(\hat{i}\,.\,\vec{a})\hat{i}+(\hat{j}\,.\,\vec{a})\hat{j}+(\hat{k}\,.\,\vec{a})\hat{k}$$
$$= 3\vec{a}-(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$$
$$= 3\vec{a}-\vec{a} = 2\vec{a}$$

21. Prove that $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\} = (\vec{b}\,.\,\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}\,.\,\vec{c})\ (\vec{a}\times\vec{d})$

Solution.

We have,

$$\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\} = \vec{a}\times\{(\vec{b}\,.\,\vec{d})\,\vec{c}-(\vec{b}\,.\,\vec{c})\,\vec{d}\}$$
$$= \vec{a}\times\{(\vec{b}\,.\,\vec{d})\,\vec{c}\}-\vec{a}\times\{(\vec{b}\,.\,\vec{c})\,\vec{d}\} \qquad \text{[by dist. law]}$$
$$= (\vec{b}\,.\,\vec{d})\,(\vec{a}\times\vec{c})-(\vec{b}\,.\,\vec{c})\,(\vec{a}\times\vec{d})$$

22. Let $\vec{a} = \alpha\hat{i}+2\hat{j}-3\hat{k}$, $\vec{b} = \hat{i}+2\alpha\hat{j}-2\hat{k}$ and $\vec{c} = 2\hat{i}-\alpha\hat{j}+\hat{k}$. Find the value(s) of a, if any, such that $\{(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\}\times(\vec{c}\times\vec{a})=0$. Find the vector product when $a=0$.

Solution.

$\{(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\}\times(\vec{c}\times\vec{a})$

$$=\left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{b}\times(\vec{c}\times\vec{a})$$
$$=\left[\vec{a}\,\vec{b}\,\vec{c}\right]\{(\vec{a}.\vec{b})\vec{c}-(\vec{b}.\vec{c})\vec{a}\}$$

which vanishes if (i) $(\vec{a}\,.\,\vec{b})\,\vec{c} = (\vec{b}\,.\,\vec{c})\,\vec{a}$ (ii) $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=0$

(i) $(\vec{a}\,.\,\vec{b})\,\vec{c}=(\vec{b}\,.\,\vec{c})\,\vec{a}$ leads to the equation $2\alpha^3+10\alpha+12=0$, $\alpha^2+6\alpha=0$ and $6\alpha-6=0$, which do not have a common solution.

(ii) $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=0$

$$\Rightarrow \begin{vmatrix} \alpha & 2 & -3 \\ 1 & 2\alpha & -2 \\ 2 & -\alpha & 1 \end{vmatrix} = 0 \Rightarrow 3\alpha = 2 \Rightarrow \alpha = \frac{2}{3}$$

when $\alpha=0$, $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=-10$, $\vec{a}\,.\,\vec{b}=6$, $\vec{b}\,.\,\vec{c}=0$ and the vector product is $-60\,(2\hat{i}+\hat{k})$.

23. If $\vec{A}+\vec{B} = \vec{a}$, $\vec{A}.\vec{a}=1$ and $\vec{A}\times\vec{B}=\vec{b}$, then prove that $\vec{A} = \dfrac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2}$ and

$$\vec{B} = \frac{\vec{b}\times\vec{a}+\vec{a}\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}.$$

Solution.

Given $\vec{A}+\vec{B}=\vec{a}$ ....(i)

$\Rightarrow \quad \vec{a}.(\vec{A}+\vec{B}) = \vec{a}.\vec{a}$

$\Rightarrow \quad \vec{a}.\vec{A}+\vec{a}.\vec{B} = \vec{a}.\vec{a}$

$\Rightarrow \quad 1+\vec{a}.\vec{B} = |\vec{a}|^2$

$\Rightarrow \quad \vec{a}.\vec{B} = |\vec{a}|^2 - 1$ (Given $\vec{A}\times\vec{B}=\vec{b}$)

$\Rightarrow \quad \vec{a}\times(\vec{A}\times\vec{B}) = \vec{a}\times\vec{b}$

$\Rightarrow \quad (\vec{a}.\vec{B})\,\vec{A}-(\vec{a}.\vec{A})\,\vec{B}=\vec{a}\times\vec{b}$

$\Rightarrow \quad (|\vec{a}|^2-1)\,\vec{A}-\vec{B} = \vec{a}\times\vec{b}$ [using equation (2)]

Solving equation (1) and (5), simultaneously, we get

$$\vec{A} = \frac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2} \text{ and } \vec{B} = \frac{\vec{b}\times\vec{a}+a\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}$$

24. Solve for $\vec{r}$, the simultaneous equations $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$, $\vec{r}.\vec{a}=0$ provided $\vec{a}$ is not perpendicular to $\vec{b}$.

Solution.

$(\vec{r}-\vec{c})\times\vec{b}=0 \Rightarrow \vec{r}-\vec{c}$ and $\vec{b}$ are collinear

$\therefore \quad \vec{r}-\vec{c}=k\vec{b}$

$\Rightarrow \quad r=\vec{c}+k\vec{b}$ ....(i)

$\Rightarrow \quad (\vec{c}+k\vec{b}).\vec{a}=0$

$\Rightarrow \quad k=-\dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}$

putting in (i) we get $\quad \vec{r}=\vec{c}-\dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\,\vec{b}$

25. If $\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$, where k is a scalar and $\vec{a},\vec{b}$ are any two vectors, then determine $\vec{x}$ in terms of $\vec{a},\vec{b}$ and k.

Solution.

$\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$ ....(i)

Premultiple the given equation vectorially by $\vec{a}$

$\vec{a}\times(\vec{x}\times\vec{a})+k(\vec{a}\times\vec{x})=\vec{a}\times\vec{b}$

$\Rightarrow \quad (\vec{a}.\vec{a})\vec{x}-(\vec{a}.\vec{x})\vec{a}+k(\vec{a}\times\vec{x})=\vec{a}\times\vec{b}$ .....(ii)

Premultiply (i) scalarly by $\vec{a}$

$[\vec{a}\ \vec{x}\ \vec{a}]+k(\vec{a}.\vec{x})=\vec{a}.\vec{b}$

$k(\vec{a}.\vec{x})=\vec{a}.\vec{b}$ ...(iii)

Substituting $\vec{x}\times\vec{a}$ from (i) and $\vec{a}.\vec{x}$ from (iii) in (ii) we get

$$\vec{x}=\frac{1}{a^2+k^2}\left[k\vec{b}+(\vec{a}\times\vec{b})+\frac{(\vec{a}.\vec{b})}{k}\vec{a}\right]$$

26. If $\vec{a}\,\vec{b}\,\vec{c}$ and $\vec{a}',\vec{b}',\vec{c}'$ be the reciprocal system of vectors, prove that

(i) $\vec{a}.\vec{a}'+\vec{b}.\vec{b}'+\vec{c}.\vec{c}'=3$ (ii) $\vec{a}\times\vec{a}'+\vec{b}\times\vec{b}'+\vec{c}\times\vec{c}'=\vec{0}$

Solution:

(i) We have : $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$

$\therefore \vec{a}.\vec{a}' + \vec{b}.\vec{b}' + \vec{c}.\vec{c}' = 1 + 1 + 1 = 3$

(ii) We have : $\vec{a}' = \lambda(\vec{b}\times\vec{c})$, $\vec{b}' = \lambda(\vec{c}\times\vec{a})$ and $\vec{c}' = \lambda(\vec{a}\times\vec{b})$, where $\lambda = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$$\vec{a}\times\vec{a}' = \vec{a}\times\lambda(\vec{b}\times\vec{c}) = \lambda\{\vec{a}\times(\vec{b}\times\vec{c})\} = \lambda\{(\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}\}$$

$$\vec{b}\times\vec{b}' = \vec{b}\times\lambda(\vec{c}\times\vec{a}) = \lambda\{\vec{b}\times(\vec{c}\times\vec{a})\} = \lambda\{(\vec{b}.\vec{a})\,\vec{c} - (\vec{b}.\vec{c})\,\vec{a}\}$$

and $\vec{c}\times\vec{c}' = \vec{c}\times\lambda(\vec{a}\times\vec{b}) = \lambda\{\vec{c}\times(\vec{a}\times\vec{b})\} = \lambda\{(\vec{c}.\vec{b})\,\vec{a} - (\vec{c}.\vec{a})\,\vec{b}\}$

$$\begin{aligned}\therefore\quad & \vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' \\ &= \lambda\,\{(\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}\} + \lambda\{(\vec{b}.\vec{a})\,\vec{c} - (\vec{b}.\vec{c})\,\vec{a}\} + \lambda\{(\vec{c}.\vec{b})\,\vec{a} - (\vec{c}.\vec{a})\,\vec{b}\} \\ &= \lambda\,[(\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c} + (\vec{b}.\vec{a})\,\vec{c} - (\vec{b}.\vec{c})\,\vec{a} + (\vec{c}.\vec{b})\,\vec{a} - (\vec{c}.\vec{a})\,\vec{b}] \\ &= \lambda\,[(\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c} + (\vec{a}.\vec{b})\,\vec{c} - (\vec{b}.\vec{c})\,\vec{a} + (\vec{b}.\vec{c})\,\vec{a} - (\vec{a}.\vec{c})\,\vec{b}] \\ &= \lambda\vec{0} = \vec{0}\end{aligned}$$

EXERCISE 1 (SUBJECTIVE)

1. Express :
   (i) The vectors $\overrightarrow{BC}$ $\overrightarrow{CA}$ and $\overrightarrow{AB}$ in terms of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$
   (ii) The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and in terms of the vectors $\overrightarrow{OC}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

Ans. (i) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

2. Given a regular hexagon *ABCDEF* with centre *O*, show that
   (i) $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$
   (ii) $\overrightarrow{OD} + \overrightarrow{OA} = 2\,\overrightarrow{OB} + \overrightarrow{OF}$
   (iii) $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{PC} = 4\,\overrightarrow{AB}$

3. The vector $-\hat{i}+\hat{j}-\hat{k}$ bisects the angle between the vectors $\vec{c}$ and $3\hat{i}+4\hat{j}$. Determine the unit vector along $\vec{c}$

Ans. $-\dfrac{1}{3}\hat{i} + \dfrac{2}{15}\hat{j} - \dfrac{14}{15}\hat{k}$

4. The sum of the two unit vectors is a unit vector. Show that the magnitude of the their difference is $\sqrt{3}$.

5. If $\vec{a}, \vec{b}$ are position vectors of the points $(1, -1)$, $(-2, m)$, find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear.

Ans. $m = 2$

6. The position vectors of the points A, B, C, D are $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$, $\hat{i}-6\hat{j}-\hat{k}$ respectively. Show that the lines *AB* and *CD* are parallel and find the ratio of their lengths.

Ans. 1 : 2

7. The vertices *P*, *Q* and *S* of a triangle *PQS* have position vectors $\vec{p}$, $\vec{q}$ and $\vec{s}$ respectively.
   (i) Find $\vec{m}$, the position vector of *M*, the mid-point of *PQ*, in terms of $\vec{p}$ and $\vec{q}$.
   (ii) Find $\vec{t}$, the position vector of *T* on *SM* such that *ST* : *TM* = 2 : 1, in terms of $\vec{p}$, $\vec{q}$ and $\vec{s}$.
   (iii) If the parallelogram *PQRS* is now completed. Express $\vec{r}$, the position vector of the point *R* in terms of $\vec{p}$, $\vec{q}$ and $\vec{s}$

   Prove that *P*, *T* and *R* are collinear.

Ans. $\vec{m} = \dfrac{1}{2}(\vec{p}+\vec{q})$, $\quad \vec{t} = \dfrac{1}{2}(\vec{p}+\vec{q}+\vec{s})$, $\quad \vec{r} = \dfrac{1}{2}(\vec{q}+\vec{p}-\vec{s})$

8. D, E, F are the mid-points of the sides BC, CA, AB respectively of a triangle. Show $\overrightarrow{FE} = 1/2\ \overrightarrow{BC}$ and that the sum of the vectors $\overrightarrow{AD}$, $\overrightarrow{BE}$, $\overrightarrow{CF}$ is zero.
9. The median AD of a triangle ABC is bisected at E and BE is produced to meet the side AC in F; show that $AF = 1/3\ AC$ and $EF = 1/4\ BF$.
10. Point L, M, N divide the sides BC, CA, AB of ΔABC in the ratios $1:4, 3:2, 3:7$ respectively. Prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, when K divides AB in the ratio $1:3$.
11. If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is angle between them, prove that $\tan\frac{\theta}{2} = \frac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$.
12. Find the values of x and y is the vectors $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpendicular vectors of equal magnitude.

Ans. $x = -\frac{31}{12}$, $y = \frac{41}{12}$

13. Let $\vec{a} = x^2\hat{i} + 2\hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = x^2\hat{i} + 5\hat{j} - 4\hat{k}$ be three vectors. Find the values of x for which the angle between $\vec{a}$ and $\vec{b}$ is acute and the angle between $\vec{b}$ and $\vec{c}$ is obtuse.

Ans. $(-3, -2) \cup (2, 3)$

14. The points O, A, B, C, D, are such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = 2\vec{a} + 3\vec{b}$, $\overrightarrow{OD} = \vec{a} + 2\vec{b}$. Given that the length of $\overrightarrow{OA}$ is three times the length of $\overrightarrow{OB}$ show that $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are perpendicular.
15. $ABCD$ is a tetrahedron and G is the centroid of the base BCD.
Prove that $AB^2 + AC^2 + AD^2 = GB^2 + GC^2 + GD^2 + 3GA^2$
16. If $\vec{p}$ and $\vec{q}$ are unit vectors forming an angle of $30°$; find the area of the parallelogram having $\vec{a} = \vec{p} + 2\vec{q}$ and $\vec{b} = 2\vec{p} + \vec{q}$ as its diagonals.

Ans. 3/4 sq. units

17. Show that $\{(\vec{a} + \vec{b} + \vec{c}) \times (\vec{c} - \vec{b})\}.\vec{a} = 2[\vec{a}\ \vec{b}\ \vec{c}]$.
18. Prove that the normal to the plane containing the three points whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ lies in the direction $\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}$
19. ABC is a triangle and EF is any straight line parallel to BC meeting AC, AB in E, F respectively. If BR and CQ be drawn parallel to AC, AB respectively to meet EF in R and Q respectively, prove that $\Delta ARB = \Delta ACQ$.
20. Show that $\vec{a}.(\vec{b} + \vec{c}) \times (\vec{a} + \vec{b} + \vec{c}) = 0$
21. One vertex of a parallelopiped is at the point $A(1, -1, -2)$ in the rectangular cartesian co-ordinate. If three adjacent vertices are at $B(-1, 0, 2)$, $C(2, -2, 3)$ and $D(4, 2, 1)$, then find the volume of the parallelopiped.

Ans. 72

22. Find the value of m such that the vectors $2\hat{i} - \hat{j} + \hat{k}$; $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + m\hat{j} + 5\hat{k}$ are coplanar.

Ans. -4

23. Show that the vector $\vec{a}, \vec{b}, \vec{c}$, are coplanar if and only if $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$, $\vec{a} + \vec{b}$ are coplanar.
24. Prove that $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$.
25. Find the unit vector coplanar with $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ and perpendicular to $\hat{i} + \hat{j} + \hat{k}$.

Ans. $\frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$ or, $\frac{1}{\sqrt{2}}(\hat{j} - \hat{k})$

26. Prove that $\vec{a}\times\{\vec{a}\times(\vec{a}\times\vec{b})\} = (\vec{a}.\vec{a})(\vec{b}\times\vec{a})$.

27. Given that $\vec{x}+\dfrac{1}{p^2}(\vec{p}\,.\,\vec{x})\,\vec{p}=\vec{q}$, show that $\vec{p}\,.\,\vec{x}=\dfrac{1}{2}\vec{p}\,.\,\vec{q}$ and find $\vec{x}$ in terms of $\vec{p}$ and $\vec{q}$.

28. If $\vec{x}\,.\,\vec{a}=0, \vec{x}\,.\,\vec{b}=0$ and $\vec{x}\,.\,\vec{c}=0$ for some non-zero vector $\vec{x}$, then show that $[\vec{a}\;\vec{b}\;\vec{c}]=0$

29. Prove that $\vec{r}=\dfrac{(\vec{r}\,.\,\vec{a})\,(\vec{b}\times\vec{c})}{[abc]}+\dfrac{(\vec{r}\,.\,\vec{b})\,(\vec{c}\times\vec{a})}{[abc]}+\dfrac{(\vec{r}\,.\,\vec{c})\,(\vec{a}\times\vec{b})}{[abc]}$

where $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors.

EXERCISE 2 (OBJECTIVE)

More than One Choice may be Correct

1. Let $\vec{a}=a_1i+a_2j+a_3k$, $\vec{b}=b_1i+b_2j+b_3k$, $\vec{c}=c_1i+c_2j+c_3k$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$,

   then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to

   (a) 0 (b) $\dfrac{1}{4}(a_1^2+a_2^2+a_3^2)\,(b_1^2+b_2^2+b_3^2)$

   (c) 1 (d) $\dfrac{3}{4}(a_1^2+a_2^2+a_3^3)\,(b_1^2+b_2^2+b_3^3)\,(c_1^2+c_2^2+c_3^3)$

2. The numbers of vectors of unit length perpendicular to vectors $\vec{a}=(1, 1, 0)$ and $\vec{b}=(0, 1, 1)$ is

   (a) one (b) two (c) three (d) infinite

3. Let $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ is of magnitude $\sqrt{\dfrac{2}{3}}$ is

   (a) $2\hat{i}+3\hat{j}-3\hat{k}$ (b) $2\hat{i}+3\hat{j}+3\hat{k}$ (c) $-2\hat{i}-\hat{j}+5\hat{k}$ (d) $2\hat{i}+\hat{j}+5\hat{k}$

4. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|\vec{c}|=\sqrt{3}$ then :

   (a) $\alpha=1, \beta=-1$ (b) $\alpha=1, \beta=\pm1$ (c) $\alpha=-1, \beta=\pm1$ (d) $\alpha=\pm1, \beta=1$

5. For three vectors $\vec{u}, \vec{v}, \vec{w}$ which of the following expressions is not equal to any of the remaining three ?

   (a) $\vec{u}.(\vec{v}\times\vec{w})$ (b) $(\vec{v}\times\vec{w}).\vec{u}$ (c) $\vec{v}(\vec{v}\times\vec{w})$ (d) $(\vec{u}\times\vec{v}).\vec{w}$

6. Which of the following expressions are meaningful ?

   (a) $\vec{u}.(\vec{v}\times\vec{w})$ (b) $(\vec{u}.\vec{v}).\vec{w}$ (c) $(\vec{u}.\vec{v})\vec{w}$ (d) $u\times(\vec{v}.\vec{w})$

7. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u}=\vec{a}-(\vec{a}.\vec{b})\vec{b}$ and $\vec{v}=\vec{a}\times\vec{b}$, then $|\vec{v}|$ is

   (a) $|\vec{u}|$ (b) $|\vec{u}|+|\vec{u}.\vec{a}|$ (c) $|\vec{u}|+|\vec{u}.\vec{b}|$ (d) $\vec{u}+\vec{u}.(\vec{a}+\vec{b})$

EXERCISE 3 (OBJECTIVE)

Only one Choice is correct

1. The scalar $\vec{A}.(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C})$ equals

   (a) 0 (b) $[\vec{A}\;\vec{B}\;\vec{C}]+[\vec{B}\;\vec{C}\;\vec{A}]$

   (b) $[\vec{A}\;\vec{B}\;\vec{C}]$ (d) none of these

2. For non-zero vectors $\vec{a}, \vec{b}, \vec{c}$, $|(\vec{a}\times\vec{b}).\vec{c}|=|\vec{a}||\vec{b}||\vec{c}|$ holds if and only if

(a) $\vec{a}.\vec{b}=0, \vec{b}.\vec{c}=0$ (b) $\vec{b}.\vec{c}=0, \vec{c}.\vec{a}=0$
(c) $\vec{c}.\vec{a}=0, \vec{a}.\vec{b}=0$ (d) $\vec{a}.\vec{b}=b\,.\,\vec{c}=\vec{c}\,.\,\vec{a}=0$

3. The volume of the parallelepiped whose sides are given by $\overline{OA}=2\hat{i}-2\hat{j}$, $\overline{OB}=\hat{i}+\hat{j}-\hat{k}$, $\overline{OC}=3\hat{i}-\hat{k}$ is
(a) $\frac{4}{13}$ (b) 2 (c) $\frac{2}{7}$ (d) none of these

4. The points with position vectors $60\hat{i}+3\hat{j}$, $40\hat{i}-8\hat{j}$, $a\hat{i}-52\hat{j}$ are collinear if
(a) $a=-40$ (b) $a=40$ (c) $a=20$ (d) none of these

5. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors and $\vec{p}, \vec{q}, \vec{r}$ are vectors defined by the relations $\vec{p}=\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{q}=\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{r}=\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, then the value of the expression
$(\vec{a}+\vec{b}).\vec{p}+(\vec{b}+\vec{c}).\vec{q}+(\vec{c}+\vec{a}).\vec{r}$ is equal to
(a) 0 (b) 1 (c) 2 (d) 3

6. Let a, b, c be distinct nonnegative numbers. If vectors $a\hat{i}+a\hat{j}+c\hat{k}$, $\hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is :
(a) the AM of a and b (b) the GM of a and b (c) the HM of a and b (d) equal to zero

7. Let $\vec{a}=\hat{i}-\hat{j}$, $\vec{b}=\hat{j}-\hat{k}$, $\vec{c}=\hat{k}-\hat{i}$. If $\vec{d}$ is a unit vector such that $\vec{a}.\vec{d}=0=(\vec{b}\;\vec{c}\;\vec{d})$, then $\vec{d}$ equals
(a) $\pm\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$ (b) $\pm\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$ (c) $\pm\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ (d) $\pm\hat{k}$

8. If $\vec{a}, \vec{b}, \vec{c}$ are non coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{1}{\sqrt{2}}(\vec{b}+\vec{c})$ then the angle between $\vec{a}$ and $\vec{b}$ is :
(a) $3\pi/4$ (b) $\pi/4$ (c) $\pi/2$ (d) π

9. Let $\vec{u}, \vec{v}, \vec{w}$ be the vectors such that $\vec{u}+\vec{v}+\vec{w}=0$, if $|\vec{u}|=3$, $|\vec{v}|=4$ and $|\vec{w}|=5$ then the value of $\vec{u}.\vec{v}+\vec{v}.\vec{w}+\vec{w}.\vec{u}$ is :
(a) 47 (b) -25 (c) 0 (d) 25

10. If $\vec{A}, \vec{B}, \vec{C}$ are three non coplanar vectors, then $(\vec{A}+\vec{B}+\vec{C}).[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]$ equals
(a) 0 (b) $[\vec{A}\vec{B}\vec{C}]$ (c) $2[\vec{A}\vec{B}\vec{C}]$ (d) $-[\vec{A}\vec{B}\vec{C}]$

11. If $\vec{p}, \vec{q}, \vec{r}$ be three mutually perpendicular vectors of the same magnitude. If a vector $\vec{x}$ satisfies the equation $\vec{p}\times((\vec{x}-\vec{q})\times\vec{p})+\vec{q}\times((\vec{x}-\vec{r})\times\vec{q})+\vec{r}\times((\vec{x}-\vec{p})\times\vec{r})=\vec{0}$, then $\vec{x}$ is given by
(a) $\frac{1}{2}(\vec{p}+\vec{q}-2\vec{r})$ (b) $\frac{1}{2}(\vec{p}+\vec{q}+\vec{r})$ (c) $\frac{1}{3}(\vec{p}+\vec{q}+\vec{r})$ (d) $\frac{1}{3}(2\vec{p}+\vec{q}-\vec{r})$

12. Let $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. If $\vec{c}$ is a vector such that $\vec{a}.\vec{c}=|\vec{c}|$, $|\vec{c}-\vec{a}|=2\sqrt{2}$ and the angle between $(\vec{a}\times\vec{b})$ and $\vec{c}$ is 30^0, then $|(\vec{a}\times\vec{b})\times\vec{c}|=$
(a) 2/3 (b) 3/2 (c) 2 (d) 3

13. Let $\vec{a}=2\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and a unit vector $\vec{c}$ be coplanar. If $\vec{c}$ perpendicular to $\vec{a}$, then $\vec{c}=$
(a) $\frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$ (b) $\frac{1}{\sqrt{2}}(-\hat{i}-\hat{j}-\hat{k})$ (c) $\frac{1}{\sqrt{5}}(\hat{i}-2\hat{j})$ (d) $\frac{1}{\sqrt{5}}(\hat{i}-\hat{j}-\hat{k})$

14. If the vectors $\vec{a}, \vec{b}, \vec{c}$ form the sides BC, CA and AB respectively of a triangle ABC, then

(a) $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=0$ (b) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

(c) $\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}$ (d) $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$

15. Let the vectors $\vec{a},\ \vec{b},\ \vec{c}$ and $\vec{d}$ be such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{a})=\vec{0}$. Let P_1 and P_2 be planes determined by the pairs of vectors $\vec{a},\ \vec{b}$ and $\vec{c}$ respectively. Then the angle between P_1 and P_2 is:

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

16. If $\vec{a},\ \vec{b},\ \vec{c}$ are unit coplanar vectors, then the scalar triple product $[2\vec{a}-\vec{b}\ \ 2\vec{b}-\vec{c}\ \ 2\vec{c}-\vec{a}]=$

(a) 0 (b) 1 (c) $-\sqrt{3}$ (d) $\sqrt{3}$

17. Let $\vec{a}=\hat{i}-\hat{k}$, $\vec{b}=x\hat{i}+\hat{j}+(1+x)\hat{k}$ and $\vec{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$. Then $[\vec{a},\vec{b},\vec{c}]$ depends on

(a) only x (b) only y (c) Neither x Nor y (d) both x and y

18. If $\vec{a},\ \vec{b},\ \vec{c}$ are unit vectors, then $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2$ does NOT exceed

(a) 4 (b) 9 (c) 8 (d) 6

19. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular to each other then the angle between $\vec{a}$ and $\vec{b}$ is

(a) 45^0 (b) 60^0 (c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\cos^{-1}\left(\frac{2}{7}\right)$

20. Let $\vec{v}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{w}=\hat{i}+3\hat{k}$. If $\vec{u}$ is a unit vector, then the maximum value of the scalar triple product $[\vec{u}\ \ \vec{v}\ \ \vec{w}]$ is

(a) -1 (b) $\sqrt{10}+\sqrt{6}$ (c) $\sqrt{59}$ (d) $\sqrt{60}$

21. The value of 'a' so that the volume of parallelopiped formed by $\hat{i}+a\hat{j}+\hat{k},\ \hat{j}+a\hat{k}$ and $a\hat{i}+\hat{k}$ becomes minimum is

(a) -3 (b) 3 (c) $1/\sqrt{3}$ (d) $\sqrt{3}$

22. If $\vec{a}=(\hat{i}+\hat{j}+\hat{k})$, $\vec{a}.\vec{b}=1$ and $\vec{a}\times\vec{b}=\hat{j}-\hat{k}$, then $\vec{b}$ is

(a) $\hat{i}-\hat{j}+\hat{k}$ (b) $2\hat{j}-\hat{k}$ (c) $\hat{i}$ (d) $2\hat{i}$

23. The unit vector which is orthogonal to the vector $3\hat{i}+2\hat{j}+6\hat{k}$ and is coplanar with the vectors $2\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$ is

(a) $\frac{2\hat{i}-6\hat{j}+\hat{k}}{\sqrt{41}}$ (b) $\frac{2\hat{i}-3\hat{j}}{\sqrt{13}}$ (c) $\frac{3\hat{i}-\hat{k}}{\sqrt{10}}$ (d) $\frac{4\hat{i}+3\hat{j}-3\hat{k}}{\sqrt{34}}$

ANSWERS

EXERCISE 2

1. c	2. b	3. a, c	4. d
5. c	6. a, c	7. a, c	

EXERCISE 3

1. a	2. d	3. d	4. a
5. d	6. b	7. a	8. a
9. C	10. d	11. b	12. b
13. a	14. b	15. a	16. a
17. c	18. b	19. b	20. c
21. c	22. c	23. c	

2

Vector Valued Function

1. DEFINITION

A vector valued function of a real variable is a rule that associates a vector $\vec{f}(t)$ with a real number t where t belongs to some subset D or R^1 called domain of $\vec{f}$. We write $\vec{f}: D \to R^3$ to denote $\vec{f}$ is a mapping of D into R^3. For example : $\vec{f}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ is a vector valued function in R^3, defined for all real numbers t. We would write $\vec{f}: R \to R^3$. At $t = 1$, the value of functions is the vector $\hat{i} + \hat{j} + \hat{k}$ which in cartesian coordinates has the terminal point (1, 1, 1).

A vector valued function of real variable can be written in component form as

$$\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

or in the form

$$\vec{f}(t) = (f_1(t),\ f_2(t),\ f_3(t))$$

Real valued functions $f_1(t), f_2(t), f_3(t)$ are called the component functions of $\vec{f}$. The first form is often used when emphasizing that $\vec{f}(t)$ is a vector and the second form is useful when considering just the terminal points of the vectors. By identifying vectors with their terminal points, a curve in space can be written as a vector valued function.

For example :

The vector function $\vec{f}(t) = (\cos t, \sin t, t)$ defines a helix in three dimensional space. As the value of t increases the terminal points of $\vec{f}(t)$ trace out a curve spiralling upward.

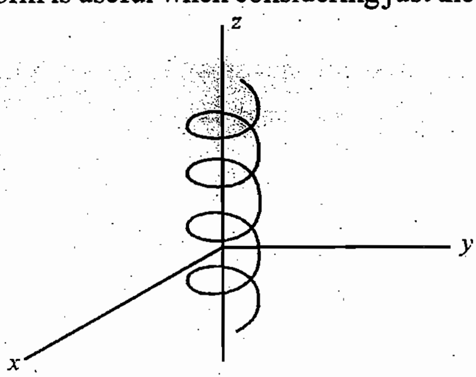


Figure 2.1

Domain of vector valued function is the intersection of domains of individual function $f_1(t), f_2(t), f_3(t)$. It is set of real values of t for which all the functions $f_1(t), f_2(t), f_3(t)$ are defined.

1. Find the domain of vector valued function

$$\vec{f}(t) = \frac{t-2}{t+2}\hat{i} + \sin t\hat{j} + \ln(9-t^2)\hat{k}$$

Solution.

The vector valued function is defined as

$$\vec{f}(t) = \frac{t-2}{t+2}\hat{i} + \sin t\hat{j} + \ln(9-t^2)\hat{k}$$

$$= \left(\frac{t-2}{t+2}, \sin t, \ln(9-t^2)\right)$$

To find the domain of $\vec{f}$, we have to find individually the domain of each component function

$$f_1(t) = \frac{t-2}{t+2}$$

$f_1(t)$ is defined for all real values of t except $t = -2$

So, domain of $f_1(t)$ is $R - \{-2\}$

$$f_2(t) = \sin t$$

$\sin t$ is defined for all real values of t.

Domain of $f_2(t)$ is R.

$$f_3(t) = \ln(9 - t^2)$$

$\ln(9 - t^2)$ is defined for $9 - t^2 > 0$

$$\Rightarrow \quad t^2 - 9 < 0$$
$$\Rightarrow \quad t^2 < 9$$
$$\Rightarrow \quad |t| < 3$$
$$-3 < t < 3$$

So, domain of $f_3(t)$ is $-3 < t < 3$.

Now, domain of vector function $\vec{f}(t)$ is intersection of domains of $f_1(t), f_2(t), f_3(t)$

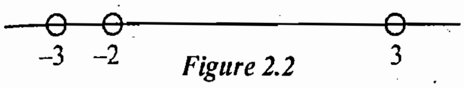


Figure 2.2

Domain of $\vec{f}(t)$ is given by

$$]-3, -2[\cup]-2, 3[$$

2. Find the domain of vector function if

$$\vec{f}(t) = \ln t\hat{i} + \frac{t-1}{(t^2-1)}\hat{j} + \sqrt{t+1}\,\hat{k}$$

Solution.

To find the domain of vector function $\vec{f}(t)$, we have to look into the value of t for which all the component functions $\ln t$, $\frac{t-1}{t^2-1}$, $\sqrt{t+1}$ are defined together

Domain of $\ln t$ is $t > 0$.

Domain of $\frac{t-1}{t^2-1}$ is $R - \{-1, 1\}$.

Domain of $\sqrt{t+1}$ is $t + 1 \geq 0$.

So, the intersection of all the individual domains is the domain of $\vec{f}(t)$

Hence, the domain of $\vec{f}(t)$ is

$$]0, 1[\cup]1, \infty[$$

2. Limit of Vector Function

Let $f(t)$ be a vector valued function. Let a be real number and $\vec{c}$ be a vector. Then we say that the limit of $\vec{f}(t)$ as t approches a equals $\vec{c}$, written as

$$\lim_{t \to a} f(t) = \vec{c} \qquad \text{if } \lim_{t \to a} \left|\vec{f}(t) - \vec{c}\right| = 0$$

If $\qquad \vec{f}(t) = (f_1(t), f_2(t), f_3(t))$

then $\qquad \lim_{t \to a} \vec{f}(t) = \left(\lim_{t \to a} f_1(t), \lim_{t \to a} f_2(t), \lim_{t \to a} f_3(t)\right)$

provided that all the limits on the right side exist.

Similarly, the continuity and the derivative of vector valued functions can also be defined in terms of its component functions.

3. Continuity of Vector Functions

Let $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ be a vector valued function and let a be a real number in its domain. Then $\vec{f}(t)$ is continuous at a if $\lim\limits_{t\to a} \vec{f}(t) = \vec{f}(a)$. Equivalently, $\vec{f}(t)$ is continuous at a if and only if $f_1(t)$, $f_2(t)$ and $f_3(t)$ are continuous at a.

4. Derivative of Vector Function

Let $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ be a vector valued function and let a be a real number in its domain. Then the derivative of $\vec{f}(t)$ at a, denoted by $\vec{f}'(a)$ is the limit of $\vec{f}'(a) = \lim\limits_{h\to 0} \dfrac{\vec{f}(a+h) - \vec{f}(a)}{h}$ if the limit exists. Equivalently $\vec{f}'(a) = (f_1'(a), f_2'(a), f_3'(a))$ if the component derivatives exists. We say that $\vec{f}(t)$ is differentiable at a if $\vec{f}'(a)$ exists.

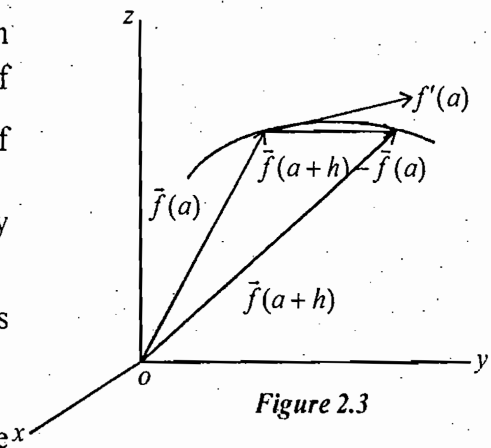


Figure 2.3

The derivative of a real valued function of a single variable derivative of a vector valued function is a tangent vector to the curve in space which function represents and it lies on the tangent line of the curve.

3. Find the limit $\lim\limits_{t\to 0} f(t)$, if

$$\vec{f}(t) = \frac{e^t - 1}{t}\hat{i} + \frac{\sqrt{1+t}-1}{t}\hat{j} + \frac{3}{1+t}\hat{k}$$

Solution.

We know that if $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$

$$\lim_{t\to 0} \vec{f}(t) = \lim_{t\to 0} \vec{f}_1(t)\hat{i} + \lim_{t\to 0} f_2(t)\hat{j} + \lim_{t\to 0} f_3(t)\hat{k}$$

$$= \lim_{t\to 0} \frac{e^t - 1}{t}\hat{i} + \lim_{t\to 0} \frac{\sqrt{1+t}-1}{t}\hat{j} + \lim_{t\to 0} \frac{3}{1+t}\hat{k}$$

$$= \lim_{t\to 0} \frac{\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots - 1\right)}{t}\hat{i} + \lim_{t\to 0} \frac{\left(\sqrt{1+t}-1\right)\left(\sqrt{1+t}+1\right)}{t\left(\sqrt{1+t}+1\right)}\hat{j} + 3\hat{k}$$

$$= \lim_{t\to 0} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\hat{i} + \lim_{t\to 0} \frac{t}{t\sqrt{1+t}+1}\hat{j} + 3\hat{k}$$

$$= \hat{i} + \frac{1}{2}\hat{j} + 3\hat{k}$$

4. Find the limit of given vector function

(a) $\lim_{t\to 4} \vec{f}(t)$ for $\vec{f}(t) = (4-t)\hat{i} + \sqrt{12+t}\,\hat{j} - \left(\cos\frac{\pi t}{8}\right)\hat{k}$

(b) $\lim_{t\to\infty} \vec{f}(t)$ for $\vec{f}(t) = \frac{\sin t}{t}\hat{i} + \frac{t+1}{3t+4}\hat{j} + \frac{\ln t^2}{t^3}\hat{k}$

Solution.

(a)
$$\lim_{t\to 4}\vec{f}(t) = \lim_{t\to 4}(4-t)\hat{i} + \lim_{t\to 4}\sqrt{12+t}\,\hat{j} - \lim_{t\to 4}\cos\frac{\pi t}{8}\hat{k}$$
$$= 0\hat{i} + 4\hat{j} - 0\hat{k}$$
$$= 4\hat{j}$$

(b)
$$\lim_{t\to\infty}\vec{f}(t) = \lim_{t\to\infty}\frac{\sin t}{t}\hat{i} + \lim_{t\to\infty}\frac{t+1}{3t+4}\hat{j} + \lim_{t\to\infty}\frac{\ln t^2}{t^3}$$

$$\lim_{t\to\infty}\frac{\sin t}{t} = 0 \qquad \text{(since } \sin t \text{ is bounded)}$$

$$\lim_{t\to\infty}\frac{t+1}{3t+4} = \lim_{t\to\infty}\frac{1+1/t}{3+4/t} = \frac{1}{3}$$

$$\lim_{t\to\infty}\frac{\ln t^2}{t^3} = \lim_{t\to\infty}\frac{\frac{1}{t^2}\cdot 2t}{3t^2} \qquad \text{(Using L'Hospital Rule)}$$
$$= \lim_{t\to\infty}\frac{2}{3t^3} = 0$$

So, $$\lim_{t\to\infty}\vec{f}(t) = \frac{1}{3}\hat{j}$$

Theorem 1: (Algebra of limits of Vector Function)

If $\vec{f}(t)$ and $\vec{g}(t)$ are vector functions of a scalar variable t, then

(i) $\lim_{t\to a}[\vec{f}(t)\pm\vec{g}(t)] = \lim_{t\to a}\vec{f}(t) \pm \lim_{t\to a}\vec{g}(t)$

(ii) $\lim_{t\to a}[\vec{f}(t)\cdot\vec{g}(t)] = \left[\lim_{t\to a}\vec{f}(t)\right]\cdot\left[\lim_{t\to a}\vec{g}(t)\right]$

(iii) $\lim_{t\to a}[\vec{f}(t)\times\vec{g}(t)] = \left[\lim_{t\to a}\vec{f}(t)\right]\times\left[\lim_{t\to a}\vec{g}(t)\right]$

Proof:

(i) Let $$\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$
and $$\vec{g}(t) = g_1(t)\hat{i} + g_2(t)\hat{j} + g_3(t)\hat{k}$$
$$\vec{f}(t)\pm\vec{g}(t) = (f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}) \pm (g_1(t)\hat{i} + g_2(t)\hat{j} + g_3(t)\hat{k})$$
$$= (f_1(t)+g_1(t))\hat{i} + (f_2(t)+g_2(t))\hat{j} + (f_3(t)+g_3(t))\hat{k}$$

We know that
$$\lim_{t\to a} f(t) = \lim_{t\to a} f_1(t)\hat{i} + \lim_{t\to u} f_2(t)\hat{j} + \lim_{t\to a} f_3(t)\hat{k}$$

So, $$\lim_{t\to a}\left(\vec{f}(t)\pm\vec{g}(t)\right) = \lim_{t\to a}(f_1(t)\pm g_1(t))\hat{i} + \lim_{t\to a}(f_2(t)\pm g_2(t)\hat{j} + \lim_{t\to a}(f_3(t)\pm g_3(t))\hat{k}$$

$$= \left(\lim_{t\to a} f_1(t) \pm \lim_{t\to a} g_1(t)\right)\hat{i} + \left(\lim_{t\to a} f_2(t) \pm \lim_{t\to a} g_2(t)\right)\hat{j} + \left(\lim_{t\to a} f_3(t) \pm \lim_{t\to a} g_3(t)\right)\hat{k}$$

(By Algebra of limit of scalar function)

(ii) $$\lim_{t\to a} (\vec{f}(t) \pm \vec{g}(t)) = \lim_{t\to a}(f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}) \cdot (g_1(t)\hat{i} + g_2(t)\hat{j} + g_3(t)\hat{k})$$

$$= \lim_{t\to a}\left(f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t)\right)$$

$$= \lim_{t\to a} f_1(t)g_1(t) + \lim_{t\to a} f_2(t)g_2(t) + \lim_{t\to a} f_3(t)g_3(t)$$

$$= \lim_{t\to a} f_1(t) \lim_{t\to a} g_1(t) + \lim_{t\to a} f_2(t) \lim_{t\to a} g_2(t) + \lim_{t\to a} f_3(t) \lim_{t\to a} g_3(t)$$

(By Algebra of limit of scalar function)

$$= \left(\lim_{t\to a} f_1(t)\hat{i} + \lim_{t\to a} f_2(t)\hat{j} + \lim_{t\to a} f_3(t)\hat{k}\right) . \left(\lim_{t\to a} g_1(t)\hat{i} + \lim_{t\to a} g_2(t)\hat{j} + \lim_{t\to a} g_3(t)\hat{k}\right)$$

$$= \lim_{t\to a} (f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}) . \lim_{t\to a} (g_1(t)\hat{i} + g_2(t)\hat{j} + g_3(t)\hat{k})$$

$$= \lim_{t\to a} \vec{f}_1(t) . \lim_{t\to a} \vec{g}_1(t)$$

(iii) $$\vec{f}(t) \times \vec{g}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix}$$

$$= (f_2(t)\, g_3(t) - f_3(t)\, g_3(t))\,\hat{i} + (f_3(t)\, g_1(t) - f_1(t)\, g_3(t))\,\hat{j} + (f_1(t)\, g_2(t) - f_2(t)\, g_1(t))\,\hat{k}$$

$$\lim_{t\to a} \vec{f}(t) \times \vec{g}(t) = \lim_{t\to a} (f_2(t)g_3(t) - f_3(t)g_2(t))\hat{i} + \cdots + \cdots$$

$$= \left(\lim_{t\to a} f_2(t) \cdot \lim_{t\to a} g_3(t) - \lim_{t\to a} f_3(t) \cdot \lim_{t\to a} g_2(t)\right)\hat{i} + \cdots + \cdots$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \lim_{t\to a} f_1(t) & \lim_{t\to a} f_2(t) & \lim_{t\to a} f_3(t) \\ \lim_{t\to a} g_1(t) & \lim_{t\to a} g_2(t) & \lim_{t\to a} g_3(t) \end{vmatrix} = \lim_{t\to a} \vec{f}(t) \times \lim_{t\to a} \vec{g}(t)$$

5. Successive Derivatives

If $\vec{f}(t)$ is a vector function of a scalar variable t, then $\frac{d\vec{f}}{dt}$ is also in general a vector function of t if $\frac{d\vec{f}}{dt}$ is differentiable, then its derivative is denoted by $\frac{d^2\vec{f}}{dt^2}$ and is called the second derivative of $\vec{f}$, similarly, the derivative of $\frac{d^2\vec{f}}{dt^2}$ is denoted by $\frac{d^3\vec{f}}{dt^3}$ and is called the third derivative of $\vec{f}$ and so on.

6. Derivative of Vector Function in terms of its components

Theorem : If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, where $f_1(t), f_2(t), f_3(t)$ are derivable function of a scalar t, then

$$\frac{d\vec{f}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

Proof : By definition

$$\frac{d\vec{f}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{f}(t+\Delta t)-\vec{f}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(f_1(t+\Delta t)\hat{i} + f_2(t+\Delta t)\hat{j} + f_3(t+\Delta t)\hat{k}) - (f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k})}{\Delta t}$$

$$= \lim_{\Delta t \to 0}\left(\frac{f_1(t+\Delta t)-f_1(t)}{\Delta t}\right)\hat{i} + \lim_{\Delta t \to 0}\left(\frac{f_2(t+\Delta t)-f_2(t)}{\Delta t}\right)\hat{j} + \lim_{\Delta t \to 0}\left(\frac{f_3(t+\Delta t)-f_3(t)}{\Delta t}\right)$$

$$= \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

7. Differentiation Formulae

Theorem : If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are differentiable vector functions of a scalar t and ϕ is a differentiable scalar function of the same variable t then

(a) $$\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

(b) $$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\cdot\vec{B}$$

(c) $$\frac{d}{dt}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$$

(d) $$\frac{d}{dt}(\phi\vec{A}) = \phi\frac{d\vec{A}}{dt} + \frac{d\phi}{dt}\vec{A}$$

(e) $$\frac{d}{dt}[\vec{A}\,\vec{B}\,\vec{C}] = \left[\frac{d\vec{A}}{dt}\,\vec{B}\,\vec{C}\right] + \left[\vec{A}\,\frac{d\vec{B}}{dt}\,\vec{C}\right] + \left[\vec{A}\,\vec{B}\,\frac{d\vec{C}}{dt}\right]$$

(f) $$\frac{d}{dt}\{\vec{A}\times(\vec{B}\times\vec{C})\} = \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C}) + \vec{A}\times\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)$$

Solution.

Let
$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$
$$\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$$

(a) $$\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d}{dt}[(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) + (B_1\hat{i} + B_2\hat{j} + B_3\hat{k})]$$

$$= \frac{d}{dt}[(A_1+B_1)\hat{i} + (A_2+B_2)\hat{j} + (A_3+B_3)\hat{k}]$$

$$= \frac{d}{dt}(A_1+B_1)\hat{i} + \frac{d}{dt}(A_2+B_2)\hat{j} + \frac{d}{dt}(A_3+B_3)\hat{k}$$

$$= \left(\frac{dA_1}{dt} + \frac{dB_1}{dt}\right)\hat{i} + \left(\frac{dA_2}{dt} + \frac{dB_2}{dt}\right)\hat{j} + \left(\frac{dA_3}{dt} + \frac{dB_3}{dt}\right)\hat{k}$$

$$= \left(\frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k}\right) + \left(\frac{dB_1}{dt}\hat{i} + \frac{dB_2}{dt}\hat{j} + \frac{dB_3}{dt}\hat{k}\right)$$

$$= \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

(b) $$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d}{dt}[A_1B_1 + A_2B_2 + A_3B_3]$$

$$= \frac{d}{dt}(A_1B_1) + \frac{d}{dt}(A_2B_2) + \frac{d}{dt}(A_3B_3)$$

$$= \left(A_1\frac{dB_1}{dt} + \frac{dA_1}{dt}B_1\right) + \left(A_2\frac{dB_2}{dt} + \frac{dA_2}{dt}B_2\right) + \left(A_3\frac{dB_3}{dt} + \frac{dA_3}{dt}B_3\right)$$

$$= \left(A_1\frac{dB_1}{dt} + A_2\frac{dB_2}{dt} + A_3\frac{dB_3}{dt}\right) + \left(\frac{dA_1}{dt}B_1 + \frac{dA_2}{dt}B_2 + \frac{dA_3}{dt}B_3\right)$$

$$= (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot\left(\frac{dB_1}{dt}\hat{i} + \frac{dB_2}{dt}\hat{j} + \frac{dB_3}{dt}\hat{k}\right)$$

$$+\left(\frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k}\right)\cdot(B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$= \vec{A}\cdot\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\cdot\vec{B}$$

(c) $$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d}{dt}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ \frac{dB_1}{dt} & \frac{dB_2}{dt} & \frac{dB_3}{dt}\end{vmatrix} + \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{dA_1}{dt} & \frac{dA_2}{dt} & \frac{dA_3}{dt}\\ B_1 & B_2 & B_3\end{vmatrix}$$

$$= \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$$

(d) $$\frac{d}{dt}(\phi\vec{A}) = \frac{d}{dt}[\phi A_1\hat{i} + \phi A_2\hat{j} + \phi A_3\hat{k}]$$

$$= \frac{d}{dt}(\phi A_1)\hat{i} + \frac{d}{dt}(\phi A_2)\hat{j} + \frac{d}{dt}(\phi A_3)\hat{k}$$

$$= \left(\phi\frac{dA_1}{dt} + \frac{d\phi}{dt}A_1\right)\hat{i} + \left(\phi\frac{dA_2}{dt} + \frac{d\phi}{dt}A_2\right)\hat{j} + \left(\phi\frac{dA_3}{dt} + \frac{d\phi}{dt}A_3\right)\hat{k}$$

$$= \phi\left(\frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k}\right) + \frac{d\phi}{dt}(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$= \phi\frac{d\vec{A}}{dt} + \frac{d\phi}{dt}\vec{A}$$

(e) $$\frac{d}{dt}[\vec{A}\,\vec{B}\,\vec{C}] = \frac{d}{dt}[\vec{A}.(\vec{B}\times\vec{C})]$$

$$= \vec{A}\cdot\frac{d}{dt}(\vec{B}\times\vec{C}) + \frac{d\vec{A}}{dt}\cdot(\vec{B}\times\vec{C}) \qquad \text{[Using (b)]}$$

$$= \vec{A}\cdot\left[\frac{d\vec{B}}{dt}\times\vec{C} + \vec{B}\times\frac{d\vec{C}}{dt}\right] + \frac{d\vec{A}}{dt}\cdot(\vec{B}\times\vec{C}) \qquad \text{[Using (c)]}$$

$$= \frac{d\vec{A}}{dt}\cdot(\vec{B}\times\vec{C})+\vec{A}\cdot\left(\frac{d\vec{B}}{dt}\times\vec{C}\right)+\vec{A}\cdot\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)$$

$$= \left[\frac{d\vec{A}}{dt}\,\vec{B}\,\vec{C}\right]+\left[\vec{A}\,\frac{d\vec{B}}{dt}\,\vec{C}\right]+\left[\vec{A}\,\vec{B}\,\frac{d\vec{C}}{dt}\right]$$

(f) $\frac{d}{dt}[\vec{A}\times(\vec{B}\times\vec{C})] = \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})+\vec{A}\times\frac{d}{dt}(\vec{B}\times\vec{C})$ [Using (c)]

$$= \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})+\vec{A}\times\left(\frac{d\vec{B}}{dt}\times\vec{C}+\vec{B}\times\frac{d\vec{C}}{dt}\right)$$

$$= \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})+\vec{A}\times\left(\frac{d\vec{B}}{dt}\times\vec{C}\right)+\vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)$$

SOLVED EXAMPLES

1. If $\vec{r}(t) = 2t^4\hat{i}+e^t\hat{j}-4t^2\hat{k}$, find $\frac{d\vec{r}}{dt}$ at $t=0$.

Solution.

$$\vec{r}(t) = 2t^4\hat{i}+e^t\hat{j}-4t^2\hat{k}$$

$$\frac{d\vec{r}}{dt} = 8t^3\hat{i}+e^t\hat{j}-8t\hat{k}$$

At $t=0$, $\frac{d\vec{r}}{dt} = \hat{j}$

2. If $\vec{r}(t) = \sin t\,\hat{i}+\cos t\,\hat{j}+t\hat{k}$. Evaluate :

(i) $\frac{d\vec{r}}{dt}$ (ii) $\frac{d^2\vec{r}}{dt^2}$ (iii) $\left|\frac{d\vec{r}}{dt}\right|$ (iv) $\left|\frac{d^2\vec{r}}{dt^2}\right|$

Solution.

$$\vec{r}(t) = \sin t\,\hat{i}+\cos t\,\hat{j}+t\hat{k}$$

$$\frac{d\vec{r}}{dt} = \cos t\,\hat{i}-\sin t\,\hat{j}+\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -\sin t\,\hat{i}-\cos t\,\hat{j}+0.\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\cos^2 t+\sin^2 t+1} = \sqrt{2}$$

$$\left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{\sin^2 t+\cos^2 t} = 1$$

3. If $\vec{r} = a\cos t\hat{i}+a\sin t\hat{j}+bt\hat{k}$, prove that

(i) $\left[\frac{d\vec{r}}{dt},\frac{d^2\vec{r}}{dt^2},\frac{d^3\vec{r}}{dt^3}\right] = a^2b$ (ii) $(\dot{\vec{r}})^2 = a^2+b^2$ (iii) $(\dot{\vec{r}}\times\ddot{\vec{r}})^2 = a^2(a^2+b^2)$

Solution.

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$$

$$\frac{d\vec{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\,\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\hat{i} - a\sin t\,\hat{j}$$

$$\frac{d^3\vec{r}}{dt^3} = a\sin t\,\hat{i} - a\cos t\,\hat{j}$$

$$\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix} = a^2 b$$

(ii) $$(\dot{\vec{r}})^2 = \dot{\vec{r}}\cdot\dot{\vec{r}}$$

$$= (-a\sin t\hat{i} + a\cos t\hat{j} + b\hat{k})\cdot(-a\sin t\hat{i} + a\cos t\hat{j} + b\hat{k})$$
$$= a^2\sin^2 t + a^2\cos^2 t + b^2$$
$$= a^2 + b^2$$

(iii) $$\dot{\vec{r}}\times\ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= ab\sin t\hat{i} - ab\cos t\hat{j} + a^2\hat{k}$$

$$(\dot{\vec{r}}\times\ddot{\vec{r}})^2 = (\dot{\vec{r}}\times\ddot{\vec{r}})\cdot(\dot{\vec{r}}\times\ddot{\vec{r}})$$
$$= a^2b^2\sin^2 t + a^2b^2\cos^2 t + a^4$$
$$= a^2b^2 + a^4$$

4. If $\vec{r} = \vec{a}\cos\alpha t + \vec{b}\sin\alpha t$. Show that

$$\vec{r}\times\frac{d\vec{r}}{dt} = \alpha\vec{a}\times\vec{b} \quad\text{and}\quad \frac{d^2\vec{r}}{dt^2} = -\alpha^2\vec{r}$$

where $\vec{a}, \vec{b}, \alpha$ are constant.

Solution.

$$\vec{r} = \vec{a}\cos\alpha t + \vec{b}\sin\alpha t$$

$$\frac{d\vec{r}}{dt} = -\alpha\vec{a}\sin\alpha t + \alpha\vec{b}\cos\alpha t$$

$$\frac{d^2\vec{r}}{dt^2} = -\alpha^2\vec{a}\cos\alpha t - \alpha^2\vec{b}\sin\alpha t$$

$$= -\alpha^2\vec{r}$$

$$\vec{r}\times\frac{d\vec{r}}{dt} = (\vec{a}\cos\alpha t + \vec{b}\sin\alpha t)\times(-\alpha\vec{a}\sin\alpha t + \alpha\vec{b}\cos\alpha t)$$

$$= \alpha\vec{a}\times\vec{b}\cos^2\alpha t - \alpha\vec{b}\times\vec{a}\sin^2\alpha t$$

$$= \alpha(\vec{a}\times\vec{b})\cos^2\alpha t + \alpha(\vec{a}\times\vec{b})\sin^2\alpha t \qquad \text{(since, } \vec{a}\times\vec{b} = -\vec{b}\times\vec{a})$$

$$= \alpha(\vec{a}\times\vec{b})$$

5. Find the first and second derivative of $\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\right]$.

Solution.

$$\frac{d}{dt}\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\right] = \left[\frac{d\bar{r}}{dt}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\right] + \left[\bar{r}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^2\bar{r}}{dt^2}\right] + \left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^3\bar{r}}{dt^3}\right]$$

$$= \left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^3\bar{r}}{dt^3}\right] \text{ as } \left\{\begin{array}{l} \left[\frac{d\bar{r}}{dt}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\right] = 0 \\ \& \left[\bar{r}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^2\bar{r}}{dt^2}\right] = 0 \end{array}\right\}$$

$$\frac{d^2}{dt^2}\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\right] = \frac{d}{dt}\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^3\bar{r}}{dt^3}\right]$$

$$= \left[\frac{d\bar{r}}{dt}\ \frac{d\bar{r}}{dt}\ \frac{d^3\bar{r}}{dt^3}\right] + \left[\bar{r}\ \frac{d^2r}{dt^2}\ \frac{d^3\bar{r}}{dt^3}\right] + \left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^4\bar{r}}{dr^4}\right]$$

$$= \left[\bar{r}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^3\bar{r}}{dt^3}\right] + \left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^4\bar{r}}{dr^4}\right]$$

6. If $\frac{d\vec{a}}{dt} = \vec{w}\times\vec{a}$ and $\frac{d\vec{b}}{dt} = \vec{w}\times\vec{b}$,show that $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{w}\times(\vec{a}\times\vec{b})$

Solution.

$$\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$$

$$= \vec{a}\times(\vec{w}\times\vec{b}) + (\vec{w}\times\vec{a})\times\vec{b}$$

$$= (\vec{a}.\vec{b})\vec{w} - (\vec{a}.\vec{w})\vec{b} + (\vec{w}.\vec{b})\vec{a} - (\vec{a}.\vec{b})\vec{w}$$

$$= (\vec{w}.\vec{b})\vec{a} - (\vec{w}.\vec{a})\vec{b}$$

$$= \vec{w}\times(\vec{a}\times\vec{b})$$

7. If $\vec{A} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\vec{B} = \sin t\hat{i} - \cos t\hat{j}$, find $\frac{d}{dt}(\vec{A}\times\vec{B})$

Solution.

We have

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= -t^3\cos t\,\hat{i} - t^3\sin t\,\hat{j} - (5t^2\cos t + t\sin t)\hat{k}$$

Hence, $$\frac{d}{dt}(\vec{A}\times\vec{B}) = -(3t^2\cos t - t^3\sin t)\hat{i} - (3t^2\sin t + t^3\cos t)\hat{j}$$

$$-(10t\cos t - 5t^2\sin t + \sin t + t\cos t)\hat{k}$$

8. Show that a vector function $\vec{f}(t)$ has a constant magnitude if and only if $\vec{f}.\frac{d\vec{f}}{dt} = 0$

Solution. The magnitude of $\vec{f}(t)$ is denoted by $\left|\vec{f}(t)\right|$

If magnitude of $\vec{f}(t)$, $\left|\vec{f}(t)\right|$ is constant

$$\Rightarrow \quad \left|\vec{f}(t)\right| = \text{constant}$$

$$\Rightarrow \quad \left|\vec{f}(t)\right|^2 = \text{constant}$$

$$\Rightarrow \quad \vec{f}(t).\vec{f}(t) = \text{constant}$$

Taking derivative on both sides

$$\vec{f}(t).\frac{d\vec{f}}{dt}(t) + \frac{d\vec{f}}{dt}(t).\vec{f}(t) = 0$$

$$\Rightarrow \quad \vec{f}.\frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt}.\vec{f} = 0$$

$$\Rightarrow \quad 2\vec{f}.\frac{d\vec{f}}{dt} = 0$$

$$\Rightarrow \quad \vec{f}.\frac{d\vec{f}}{dt} = 0$$

Now, let us prove its converse

let $\quad \vec{f}.\dfrac{d\vec{f}}{dt} = 0$

$$\Rightarrow \quad \frac{1}{2}\frac{d}{dt}\left(\vec{f}.\vec{f}\right) = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left|\vec{f}\right|^2 = 0$$

$$\Rightarrow \quad \left|\vec{f}\right|^2 = \text{constant}$$

$$\Rightarrow \quad \left|\vec{f}\right| = \text{constant}$$

So, a vector function $\vec{f}(t)$ has a constant magnitude if and only if $\vec{f}.\dfrac{d\vec{f}}{dt} = 0$

9. Show that a necessary and sufficient condition for a vector for $\vec{f}(t)$ to have constant direction is

$$\vec{f} \times \frac{d\vec{f}}{dt} = 0$$

Solution. $\quad \vec{f}(t) = \phi(t)\hat{F}(t)$

Where $\phi(t)$ denotes the magnitude of $\vec{f}(t)$ and $\hat{F}(t)$ is a unit vector in the direction of $\vec{f}(t)$

$$\frac{d\vec{f}}{dt} = \frac{d}{dt}(\phi\hat{F}) = \frac{d\phi}{dt}\hat{F} + \phi\frac{d\hat{F}}{dt}$$

$$\vec{f} \times \frac{d\vec{f}}{dt} = \phi\hat{F} \times \left(\frac{d\phi}{dt}\hat{F} + \phi\frac{d\hat{F}}{dt}\right)$$

$$= \frac{d\phi}{dt}\hat{F} \times \hat{F} + \phi^2\hat{F} \times \frac{d\hat{F}}{dt}$$

$$= \phi^2\hat{F} \times \frac{d\hat{F}}{dt} \qquad (\because \hat{F} \times \hat{F} = 0)$$

Condition is necessary

Suppose that the direction of $\vec{f}$ is constant, then $\hat{F}$ is a constant vector.

So, $$\frac{d\hat{F}}{dt} = 0$$

So, $$\vec{f} \times \frac{d\vec{f}}{dt} = 0$$

Condition is sufficient

Let $$\vec{f} \times \frac{d\vec{f}}{dt} = 0$$

If $\vec{f} = \phi\hat{F}$ where $\hat{F}$ is a unit vector.

Then $\vec{f} \times \frac{d\vec{f}}{dt} = 0$ reduced to $\hat{F} \times \frac{d\hat{F}}{dt} = 0$

Since, $\hat{F}$ has constant magnitude

So, $$\hat{F} \times \frac{d\hat{F}}{dt} = 0$$

Since, $\hat{F} \times \frac{d\hat{F}}{dt} = 0$ & $\hat{F}.\frac{d\hat{F}}{dt} = 0$

Hence, $\frac{d\hat{F}}{dt} = 0$ ie. $\hat{F}$ is a constant vector

Hence, the direction of $\vec{f}$ remains the same

10. Find a unit tangent vector to any point on the curve $x = a\cos wt, y = a\sin wt, z = bt$ when a, b, w are constant.

Solution.

Any point on the curve is given by the vector

$$\vec{r} = a\cos wt\,\hat{i} + a\sin wt\,\hat{j} + bt\hat{k}$$

A tangent vector to the given curve at any point

$$\frac{d\vec{r}}{dt} = -aw\sin wt\,\hat{i} + aw\cos wt\,\hat{j} + b\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{a^2w^2\sin^2 wt + a^2w^2\cos^2 wt + b^2}$$

$$= \sqrt{a^2w^2 + b^2}$$

The unit tangent vector is given by

$$\frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{-aw\sin wt\hat{i} + aw\cos wt\hat{j} + b\hat{k}}{\sqrt{a^2w^2 + b^2}}$$

11. If $\vec{r} = \cos nt\,\hat{i} + \sin nt\,\hat{j}$, where n is a constant and t varies, show that $\vec{r} \times \frac{d\vec{r}}{dt} = n\hat{k}$

Solution.

$$\vec{r} = \cos nt\,\hat{i} + \sin nt\,\hat{i}$$

$$\frac{d\vec{r}}{dt} = -n\sin nt\,\hat{i} + n\cos nt\,\hat{j}$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos nt & \sin nt & 0 \\ -n\sin nt & n\cos nt & 0 \end{vmatrix} = n\hat{k}$$

12. If $\vec{r}$ is the position vector of a moving point and r is the magnitude of $\vec{r}$, show that

$$\vec{r}.\frac{d\vec{r}}{dt} = r\frac{dr}{dt}$$

Interpret the $\vec{r}.\frac{d\vec{r}}{dt}$ and $\vec{r}\times\frac{d\vec{r}}{dt} = 0$

Solution.

The position of moving point is described by vector $\vec{r}$

$$\vec{r} = r\hat{e}_r$$

where $\hat{e}_r$ is a unit vector in the direction of r

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}.\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

So,
$$\vec{r}.\frac{d\vec{r}}{dt} = r\hat{e}_r.\left(\frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt}\right)$$

$$= r\frac{dr}{dt}\hat{e}_r.\hat{e}_r + r^2\hat{e}_r.\frac{d\hat{e}_r}{dt}$$

$$= r\frac{dr}{dt} \text{ (Since, } \hat{e}_r \text{ is a vector of constant magnitude, So, } \hat{e}_r.\frac{d\hat{e}_r}{dt} = 0\text{)}$$

$\vec{r}.\frac{d\vec{r}}{dt} = 0$ implies $\frac{d\vec{r}}{dt}$ lies perpendicular to $\vec{r}$

$\vec{r}\times\frac{d\vec{r}}{dt} = 0$ implies $\frac{d\vec{r}}{dt}$ lies parallel to $\vec{r}$

Since, both are holding together, but $\frac{d\vec{r}}{dt}$ cannot be perpendicular as well as parallel to $\vec{r}$

So, $\frac{d\vec{r}}{dt} = 0$, hence, $\vec{r}$ is a constant vector

13. If $\vec{r}$ a unit vector, then prove that $\left|\vec{r}\times\frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$.

Solution.

$\vec{r}$ is a unit vector (constant magnitude)

So,
$$\vec{r}.\frac{d\vec{r}}{dt} = 0$$

Hence, $\frac{d\vec{r}}{dt}$ is perpendicular to $\vec{r}$

$$\vec{r}\times\frac{d\vec{r}}{dt} = |\vec{r}|\left|\frac{d\vec{r}}{dt}\right|.\sin 90°\,\hat{e}_n = \left|\frac{d\vec{r}}{dt}\right|\hat{e}_n$$

where $\hat{e}_n$ is unit vector normal to plane containing $\vec{r}$ & $\frac{d\vec{r}}{dt}$

So, $$\left|\vec{r}\times\frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$$

14. If $\vec{r} = t^3\hat{i} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{j}$, show that $\vec{r}\times\frac{d\vec{r}}{dt} = \hat{k}$.

Solution

$$\vec{r} = t^3\hat{i} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{j}$$

$$\frac{d\vec{r}}{dt} = 3t^2\hat{i} + \left(6t^2 + \frac{2}{5t^3}\right)\hat{j}$$

$$\vec{r}\times\frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^3 & 2t^3 - \frac{1}{5t^2} & 0 \\ 3t^2 & 6t^2 + \frac{2}{5t^3} & 0 \end{vmatrix}$$

$$= \left(6t^5 + \frac{3}{5} - 6t^5 + \frac{3}{5}\right)\hat{k}$$

$$= \hat{k}$$

15. If $\vec{r} = e^{nt}\vec{a} + e^{-nt}\vec{b}$ when $\vec{a}$ and $\vec{b}$ are constant vectors, show that $\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$

Solution.

$$\vec{r} = e^{nt}\vec{a} + e^{-nt}\vec{b}$$

$$\frac{d\vec{r}}{dt} = ne^{nt}\vec{a} - ne^{-nt}\vec{b}$$

$$\frac{d^2\vec{r}}{dt^2} = n^2e^{nt}\vec{a} + n^2e^{-nt}\vec{b}$$

$$= n^2\vec{r}$$

So, $$\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$$

16. If $\vec{r} = \vec{a}\sin wt + \vec{b}\cos wt + \frac{\vec{c}t}{w^2}\sin wt$, prove that

$$\frac{d^2\vec{r}}{dt^2} + w^2\vec{r} = \frac{2\vec{c}}{w}\cos wt$$

where $\vec{a}, \vec{b}, \vec{c}$ are constant vector and w is a constant scalar.

Solution.

$$\vec{r} = \vec{a}\sin wt + \vec{b}\cos wt + \frac{\vec{c}t}{w^2}\sin wt$$

$$\frac{d\vec{r}}{dt} = \vec{a}w\cos wt - \vec{b}w\sin wt + \frac{\vec{c}}{w^2}\sin wt + \frac{\vec{c}t}{w}\cos wt$$

$$\frac{d^2\vec{r}}{dt^2} = -\vec{a}w^2\sin wt - \vec{b}w^2\cos wt + \frac{\vec{c}}{w}\cos wt + \frac{\vec{c}}{w}\cos wt - \vec{c}t\sin wt$$

$$\frac{d^2\vec{r}}{dt^2}+w^2\vec{r} = \frac{2\vec{c}}{w}\cos wt$$

17. Show that $\vec{r}=\vec{a}e^{mt}+\vec{b}e^{nt}$ is the solution of the differential equation $\frac{d^2\vec{r}}{dt^2}-(m+n)\frac{d\vec{r}}{dt}+mn\vec{r}=0$

Hence, solve $\frac{d^2\vec{r}}{dt^2}-\frac{d\vec{r}}{dt}-2\vec{r}=0$ where $\vec{r}=\hat{i}$ and $\frac{d\vec{r}}{dt}=\hat{j}$ for $t=0$

Solution.

$$\vec{r} = \vec{a}e^{mt}+\vec{b}e^{nt}$$

$$\frac{d\vec{r}}{dt} = m\vec{a}e^{mt}+n\vec{b}e^{nt}$$

$$\frac{d^2\vec{r}}{dt^2} = m^2\vec{a}e^{mt}+n^2\vec{b}e^{nt}$$

$$\frac{d^2\vec{r}}{dt^2}-(m+n)\frac{d\vec{r}}{dt}+mn\vec{r} = m^2\vec{a}e^{mt}+n^2\vec{b}e^{nt}-(m+n)(m\vec{a}e^{mt}+n\vec{b}e^{nt})+mn(\vec{a}e^{mt}+\vec{b}e^{nt}) = 0$$

putting $m=2, n=1$

$$\vec{r} = \vec{a}e^{2t}+\vec{b}e^{-t} \text{ is solution of } \frac{d^2\vec{r}}{dt^2}-\frac{d\vec{r}}{dt}-2\vec{r}=0$$

$$\frac{d\vec{r}}{dt} = 2\vec{a}e^{2t}-\vec{b}e^{-t}$$

At $t=0$, $\vec{r}=\hat{i}$

$$\Rightarrow \quad \vec{a}+\vec{b} = \hat{i} \qquad \text{(i)}$$

At $t=0$, $\frac{d\vec{r}}{dt} = \hat{j}$

$$\Rightarrow \quad 2\vec{a}-\vec{b} = \hat{j} \qquad \text{(ii)}$$

Solving (i) & (ii)

$$\vec{a} = \frac{1}{3}(\hat{i}+\hat{j}) \ \& \ \vec{b} = \frac{1}{3}(2\hat{i}-\hat{j})$$

So,
$$\vec{r} = \frac{1}{3}(\hat{i}+\hat{j})e^{2t}+\frac{1}{3}(2\hat{i}-\hat{j})e^{-t}$$
$$= \frac{1}{3}(e^{2t}+2e^{-t})\hat{i}+\frac{1}{3}(e^{2t}-e^{-t})\hat{j}$$

18. If $\hat{R}$ be a unit Vector in the direction of $\vec{r}$, prove that $\hat{R}\times\frac{d\hat{R}}{dt}=\frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt}$ where $r=|\vec{r}|$

Solution.

The vector $\vec{r}$ can be written as

$$\vec{r} = r\hat{R}$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{R}+r\frac{d\hat{R}}{dt}$$

$$\vec{r}\times\frac{d\vec{r}}{dt} = r\hat{R}\times\left(\frac{dr}{dt}\hat{R}+r\frac{d\hat{R}}{dt}\right)$$

$$= r\frac{dr}{dt}\hat{R}\times\hat{R}+r^2\hat{R}\times\frac{d\hat{R}}{dt}$$

$$= r^2\hat{R}\times\frac{d\hat{R}}{dt}$$

Hence, $\hat{R}\times\frac{d\hat{R}}{dt} = \frac{1}{r^2}\bar{r}\times\frac{d\bar{r}}{dt}$

19. Show that if $\bar{a},\bar{b},\bar{c}$ are constant vectors, then $\bar{r} = \bar{a}t^2+\bar{b}t+\bar{c}$ is the path of a particle moving **with** constant acceleration.

Solution. The position of particle is described by

$$\bar{r} = \bar{a}t^2+\bar{b}t+\bar{c}$$

Velocity, $\bar{v} = \frac{d\bar{r}}{dt} = 2t\bar{a}+\bar{b}$

The acceleration of particle $\frac{d^2\bar{r}}{dt^2} = \frac{d\bar{v}}{dt} = 2\bar{a}$

Hence, the particle is moving with a constant acceleration.

20. A particle moves the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \pi/2$. Find also the magnitudes of the velocity and acceleration at any time t.

Solution. Let $\bar{r}$ is the position vector of the particle at time t

$$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$$
$$= 4\cos t\,\hat{i}+4\sin t\,\hat{j}+6t\,\hat{k}$$

Velocity of particle is given by $\bar{v} = \frac{d\bar{r}}{dt} = -4\sin t\,\hat{i}+4\cos t\,\hat{j}+6\hat{k}$

The acceleration of particle is given by

$$\bar{a} = \frac{d\bar{v}}{dt} = -4\cos t\,\hat{i}-4\sin t\,\hat{j}$$

Magnitude of velocity at time t

$$v = |\bar{v}| = \sqrt{16\sin^2 t+16\cos^2 t+36}$$
$$= \sqrt{52} = 2\sqrt{13}$$

Magnitude of acceleration

$$a = |\bar{a}| = \sqrt{16\cos^2 t+16\sin^2 t} = 4$$

At $t = 0$, $\bar{v} = 4\hat{j}+6\hat{k}$

$\bar{a} = -4\hat{i}$

At $t = \frac{\pi}{2}$, $\bar{v} = -4\hat{i}+6\hat{k}$

$\bar{a} = -4\hat{j}$

21. A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = 2t+5$ where t is the time. Find the component of its velocity and acceleration at $t = 1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$.

Solution. Let $\bar{r}$ be the position vector of point (x, y, z) on the curve along which particle is moving.

$$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k} = (t^3+1)\hat{i}+t^2\hat{j}+(2t+5)\hat{k}$$

$$\text{Velocity, } \vec{v} = \frac{d\vec{r}}{dt} = 3t^2\hat{i} + 2t\hat{j} + 2\hat{k}$$

Acceleration $\quad \vec{a} = \dfrac{d\vec{v}}{dt} = 6t\,\hat{i} + 2\,\hat{j}$

At time $t=1$ $\quad \vec{v} = 3\hat{i} + 2\hat{j} + 2\hat{k}$

$$\vec{a} = 6\hat{i} + 2\hat{j}$$

Unit vector in direction of $\hat{i} + \hat{j} + 3\hat{k}$

$$\hat{n} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}$$

The component of velocity in direction of $\hat{n}$ is given by

$$\vec{v}.\hat{n} = (3\hat{i} + 2\hat{j} + 2\hat{k}).\frac{(\hat{i} + \hat{j} + 3\hat{k})}{\sqrt{11}}$$

$$= \frac{11}{\sqrt{11}} = \sqrt{11}$$

The component of acceleration in direction of $\hat{n}$ is given by

$$\vec{a}.\hat{n} = (6\hat{i} + 2\hat{j}).\frac{(\hat{i} + \hat{j} + 3\hat{k})}{\sqrt{11}}$$

$$= \frac{8}{\sqrt{11}}$$

22. A particle moves so that it position vector is given by $\vec{r} = \cos wt\,\hat{i} + \sin wt\,\hat{j}$, when w is a constant show that

(a) The velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$

(b) The acceleration $\vec{a}$ is directed towards the origin and has magnitude proportional to the distance from the origin

(c) $\vec{r} \times \vec{v}$ = a constant vector

Solution:

(a) The position vector of particle is given by

$$\vec{r} = \cos wt\,\hat{i} + \sin wt\,\hat{j}$$

Velocity of particle

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= -w\sin wt\,\hat{i} + w\cos wt\,\hat{j}$$

$$\vec{r}.\vec{v} = (\cos wt\,\hat{i} + \sin wt\,\hat{j}).(-w\sin wt\,\hat{i} + w\cos wt\,\hat{j})$$

$$= 0$$

Since, $\quad \vec{r}.\vec{v} = 0$, So, velocity $\vec{v}$ is perpendicular to $\vec{r}$

(b) Acceleration $\vec{a}$ of the particle is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = -w^2\cos wt\,\hat{i} - w^2\sin wt\,\hat{j}$$

$$= -w^2\vec{r}$$

Hence, the acceleration $\vec{a}$ is directed towards the origin and has magnitude proportional to the distance from the origin.

(c) $$\bar{r}\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos wt & \sin wt & 0 \\ -w\sin wt & w\cos wt & 0 \end{vmatrix}$$

$$= w\hat{k}$$

So, $\bar{r}\times\bar{v}$ = constant vector

EXERCISE

1. $\vec{F},\vec{G},\vec{H}$, are differentiable vector function of a scalar variable t and if

$$\frac{d\vec{F}}{dt} = \vec{H}\times\vec{F} \text{ and } \frac{d\vec{G}}{dt} = \vec{H}\times\vec{F}$$

Show that $\frac{d}{dt}(\vec{F}\times\vec{G}) = \vec{H}\times(\vec{F}\times\vec{G})$

2. If $\vec{F},\vec{G}$ and $\vec{H}$ are differentiable vector function of scalar variable t, then show that

(i) $$\frac{d}{dt}(\vec{F}.(\vec{G}\times\vec{H})) = \frac{d\vec{F}}{dt}.(\vec{G}\times\vec{H}) + \vec{F}.\left(\frac{d\vec{G}}{dt}\times\vec{H}\right) + F.\left(\vec{G}\times\frac{d\vec{H}}{dt}\right)$$

(ii) $$\frac{d}{dt}(\vec{F}\times(\vec{G}\times\vec{H})) = \frac{d\vec{F}}{dt}\times(\vec{G}\times\vec{H}) + \vec{F}\times\left(\frac{d\vec{G}}{dt}\times\vec{H}\right) + \vec{F}\times\left(\vec{G}\times\frac{d\vec{H}}{dt}\right)$$

3. If $\vec{f}(t)$ and $\vec{g}(t)$ are two derivable function of t, then prove that

$$\frac{d}{dt}(\vec{f}(t).\vec{g}(t)) = \vec{f}(t).\frac{d}{dt}\vec{g}(t) + \frac{d\vec{f}(t)}{dt}.\vec{g}(t)$$

4. If $\vec{f},\vec{g},\vec{h}$ are derivable function of t, prove that

$$\frac{d}{dt}[\vec{f}\ \vec{g}\ \vec{h}] = \left[\frac{d\vec{f}}{dt}\ \vec{g}\ \vec{h}\right] + \left[\vec{f}\ \frac{d\vec{g}}{dt}\ \vec{h}\right] + \left[\vec{f}\ \vec{g}\ \frac{d\vec{h}}{dt}\right]$$

5. A particle moves along the curve $x=2t^2, y=t^2-4t, z=3t-5$ where t is the time. Find the components of its velocity and acceleration at $t=1$ in the direction $\hat{i}-3\hat{j}+2\hat{k}$. **Ans.** $\left(\frac{16}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\right)$

6. A particle moves along the curve

$$x = 2t^2,\ y = t^2 - 4t,\ z = 3t - 5$$

Where t is time. Find the components of the velocity and acceleration at time $t = 1$ in the direction $2\hat{i}+2\hat{j}+\hat{k}$. (**Ans.** $\frac{7}{3}$, 4)

7. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.

(i) Determine its velocity and acceleration at t time.

Ans. (i) $\vec{v} = -e^{-t}\hat{i} - 6\sin 3t\hat{j} + 6\cos 3t\hat{k}$, $\vec{a} = e^{-t}\hat{i} - 18\cos 3t\hat{j} - 18\sin 3t\hat{k}$

(ii) Find the magnitude of the velocity and acceleration at $t = 0$ **Ans.** $\vec{v} = -\hat{i} + 6\hat{k}$, $\vec{a} = \hat{i} - 18\hat{j}$

8. If $\vec{a} = \sin\theta\hat{i} + \cos\theta\hat{j} + \theta\hat{k}$, $\vec{b} = \cos\theta\hat{i} - \sin\theta\hat{j} - 3\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} - 3\hat{k}$, find

$\left(\frac{d}{d\theta}\{\vec{a}\times(\vec{b}\times\vec{c})\}\right)$ at $\theta = \pi/2$ **Ans.** $6\hat{i} + 15\hat{j} + 9\hat{k}$

3

GRADIENT, DIVERGENCE AND CURL

3.1 DIRECTIONAL DERIVATIVES

Suppose $f(x)$ is a function of single variable x. What does the derivative $\frac{df}{dx}$ signify? The answer is that the derivative $\frac{df}{dx}$ tells us how the function $f(x)$ varies when the independent variable x changes. If $\frac{df}{dx}$ is positive, it means the function $f(x)$ increases with increase in x and decreases with decrease in x as shown in Figure 3.1 (a). If $\frac{df}{dx}$ is negative, then the function decreases with increase in x & vice versa as shown in Figure 3.1 (b)

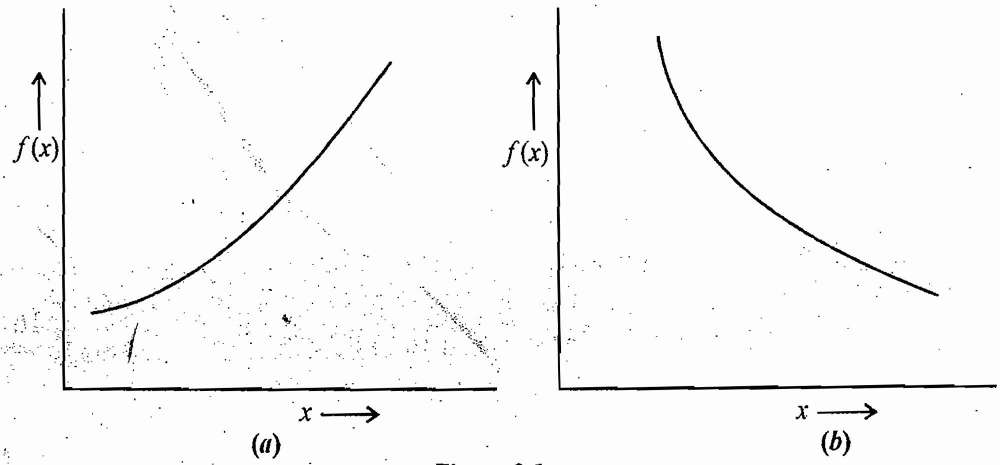


Figure 3.1

For the function plotted in Figure 3.1 (a), $\frac{df}{dx}$ is positive for every points & for the function plotted in Figure 3.1(b), $\frac{df}{dx}$ is negative at every point x.

The magnitude of $\frac{df}{dx}$ defines the magnitude of variation

The magnitude of $\frac{df}{dx}$ at A is less than that of $\frac{df}{dx}$ at B.

Geometrically, $\frac{df}{dx}$ at any point measures the slope of tangent to the curve at that point.

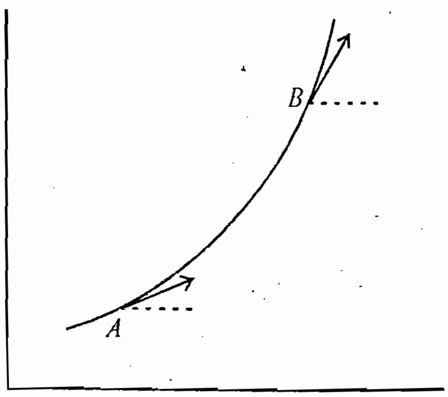


Figure 3.2

Now, let us discuss the concept of derivative for the function of two variables.

Let $z = f(x, y)$ is a function of two variable x & y

Now a derivative is supposed to tell us how fast the function varies, if we move a little distance in the domain. The domain of $f(x, y)$ is part of xy plane or whole xy plane.

For example if $z = x^2 + y^2 = f(x, y)$

The domain of $f(x, y)$ is whole xy plane

The situation now is complicated because one can move in several direction since the domain is two dimensional unlike the previous case where the domain was part of x axis or whole x axis.

Here we are interested in finding how does f varies if we move a little distance in the domain.

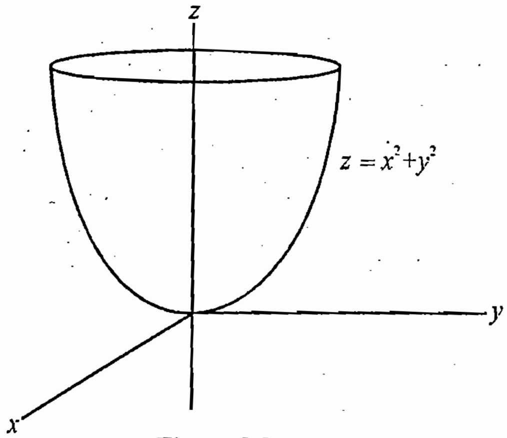


Figure 3.3

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

The differential df denotes the change in function f when we move a distance dr such that change in x value is dx and y value is dy.

Actually, we are moving in $\overline{dr}$ direction such that

$$\overline{dr} = dx\,\hat{i} + dy\,\hat{j}$$

So, change in f as we moves by distance $\overline{dr}$

$$df = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot \overline{dr}$$

In the domain of function of two variable, $\overline{dr}$ can have infinite possibilities since, there are infinite directions along which one can move.

If we moves along a curve C in the domain such that

$$x = \phi_1(t) \ \& \ y = \phi_2(t)$$

The change in f as we move along the curve C is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\left(\frac{dx}{dt}\hat{i} + \frac{\partial y}{\partial t}\hat{j}\right)$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\frac{d\bar{r}}{dt}$$

$\frac{d\bar{r}}{dt}$ is the unit vector in the direction of tangent to the curve C.

So, the derivative of f in the direction of $\frac{d\bar{r}}{dt}$ is given by

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\frac{d\bar{r}}{dt} = \nabla f\cdot\hat{a}$$

So, $\frac{df}{dt}$ is called directional derivative

$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}$ is called gradient of function f

If $\hat{a}$ is a unit vector in the direction along which we want to find the variation of f.

If $\hat{a} = \hat{i}$, then $\dfrac{df}{dt} = \nabla f \cdot \hat{i} = \dfrac{\partial f}{\partial x}$

So, partial derivative $\dfrac{\partial f}{\partial x}$ denotes the rate of change in f as one moves along the x axis or parallel to x axis.

If $\hat{a} = \hat{j}$, then $\dfrac{df}{dt} = \nabla f \cdot \hat{j} = \dfrac{\partial f}{\partial y}$

So, partial derivative $\dfrac{\partial f}{\partial y}$ denotes the rate of change in f as one moves along y axis or parallel to y axis.

So, partial derivatives are a special kinds of directional derivative.

Let us interpret the partial derivative geometrically,

when we find $\dfrac{\partial f}{\partial x}$ or $\dfrac{\partial z}{\partial x}$. We keep y = constant.

y = constant geometrically represents a plane parallel to x axis.

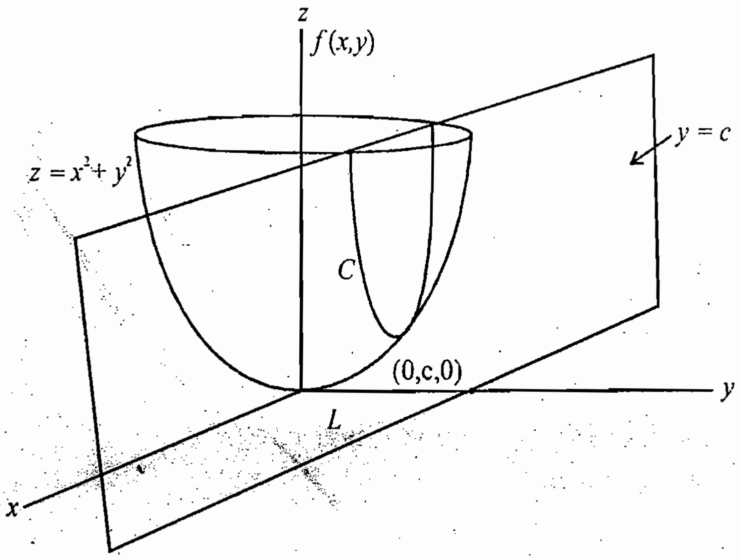


Figure 3.4

So, $z = x^2 + y^2$ & $y = c$ is together satisfied by the points which lie on the curve of intersection C of paraboloid $z = x^2 + y^2$ and plane $y = c$ as shown in Figure 3.4.

The curve C represents the variation in $f(x,y)$ as one moves along line L in the domain of $f(x, y)$ i.e. xy–plane.

Actually, $\dfrac{\partial f}{\partial x} = \left.\dfrac{df}{dx}\right|_{y=\text{const}}$ means $\dfrac{\partial f}{\partial x}$ is the slope of tangent to the curve of intersection of the surface $z = f(x,y)$ and $y = c$.

Similarly $\dfrac{\partial f}{\partial y} = \left.\dfrac{df}{dy}\right|_{x=\text{const}}$ means $\dfrac{\partial f}{\partial y}$ is the slope of tangent to the curve of intersection of the surface $z = f(x, y)$ and $x = c$.

So, if one want to find how f varies as one moves in the direction of $\hat{a}$ in domain of function.

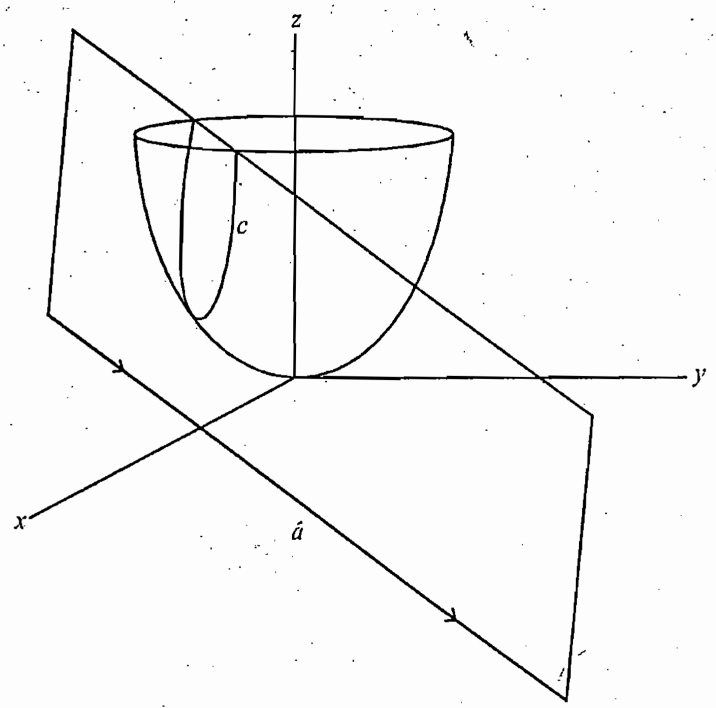


Figure 3.5

Let us draw a plane containing vector $\hat{a}$ and perpendicular to xy plane. The curve of intersection C of this plane and the $z = f(x, y)$ denotes the variation of f as one moves along the direction specified by $\hat{a}$. The directional derivative denotes the slope of tangent to the curve of intersection C.

Mathematically, in general, the directional derivative can be found by $\nabla f \cdot \hat{a}$ where $\hat{a}$ is the unit vector in the direction along which we want to find the directional derivative. Variation of z in different direction can also be studied through level curves.

The family of level curves is described by the equation.

$$f(x, y) = c \text{ where } c \text{ is arbitrary constant}$$

You put value of c, you will get different member of the family.

Suppose $z = f(x, y) = x^2 + y^2$ is a paraboloid.

The variation of z can be studied by drawing family of level curves given by $f(x, y) = c$ ie. $x^2 + y^2 = c$.

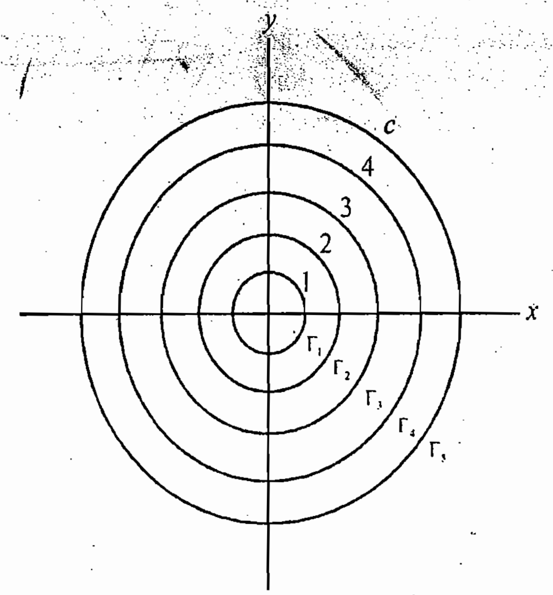


Figure 3.6

Each level curve denotes the set of points in domain ie. xy plane for which the value of $f(x, y)$ is constant.

For the given functions $f(x.y) = x^2 + y^2$, the family of level curves is the family of circles $x^2 + y^2 = c$ as shown in Figure $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \ldots \Gamma_c$ are the curves which represents set of points for which the value of z is equal to 1, 2, 3, 4, ...c.

The directional derivative of f in direction defined by $\hat{a}$ is given by

$$\nabla f \cdot \hat{a} = |\nabla f| \cos\theta$$

The directional derivative along the level curve ie. in direction of tangent to level curve is zero as z is constant on level curve.

So, if $\hat{a}$ is a unit vector along tangent is level curve

$$\nabla f \cdot \hat{a} = 0$$

It means ∇f is perpendicular to the tangent to the level curve $f(x, y) = c$.

Also, $\nabla f \cdot \hat{a} = |\nabla f| \cos\theta$ is maximum if $\theta = 0$ which means the directional derivative is maximum in the direction of ∇f

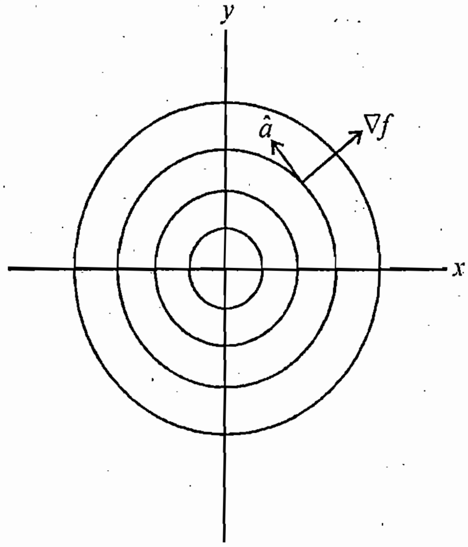


Figure 3.7

So, the properties of gradient ∇f can be summerised as

(a) The directional derivative of f in the direction described by unit vector $\hat{a}$ is equal to $\nabla f \cdot \hat{a}$

(b) The gradient lies in the direction normal to the level curve.

(c) The gradient ∇f lies in the direction in which the directional derivative is maximum.

The concept of directional derivative and gradient can be extended to function of three variables.

For the function of three variables, the domain of function $f(x, y, z)$ is a region in three dimensional space.

For example $f(x, y, z) = x^2 + y^2 - z$ is function of three variables.

The domain of function f is R^3. The directional derivative of function f in the direction defined a unit vector $\hat{a}$ is equal to $\nabla f \cdot \hat{a}$. The variation of function f can be studied by the family of level surfaces given by $f(x, y, z) = c$.

$f(x, y, z) = x^2 + y^2 - z = c$ denotes family of paraboloids.

$$z + c = x^2 + y^2$$

where c is arbitrary constants.

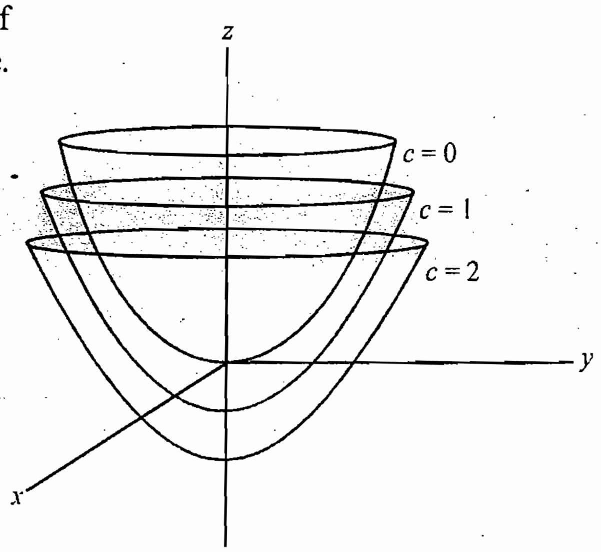


Figure 3.8

The level surfaces denotes set of points in domain which is a region in three dimensional space or whole R^3 for which $f(x, y, z)$ is constant.

The directional derivative of f along the direction of unit vector $\hat{a} = \nabla f \cdot \hat{a}$.

Since, f is constant on the level surface, so, directional derivative in the direction tangential to level

the directional derivative at any point on the level surface in the direction $\hat{a}$ lying in the tangent plane at that point is zero.

$$\nabla f \cdot \hat{a} = 0$$

So, ∇f at any point on the level surface lies normal to the tangent plane at that point.

Also, the directional derivative $\nabla f \cdot \hat{a}$ is maximum in the direction $\hat{a}$ along ∇f. So, the gradient ∇f lies in the direction in which the directional derivative is maximum.

The above discussion on directional derivative can be summarised as follows:

Definition: Let $f(x, y, z)$ define a scalar field in a region R and let P be any point in this region. Suppose Q is a point in this region. In the neighbourhood of P in the direction of a given unit vector $\hat{a}$.

Then $\lim_{Q \to P} \frac{f(Q) - f(P)}{PQ}$ if it exists, is called the directional derivative of f at P in the direction of $\hat{a}$.

Theorem 1: The directional derivative of scalar field f at a point $P(x, y, z)$ in the direction of unit vector $\hat{a}$ is given by

$$\frac{df}{ds} = \nabla \cdot \hat{a}$$

Theorem 2: If $\hat{n}$ be a unit vector normal to the level surface $f(x, y, z) = c$ at a point $P(x, y, z)$ and n be the distance of P from some fixed point A in the direction of $\hat{n}$ so that δn represents element of normal at P in the direction of $\hat{n}$ then grad $f = \frac{df}{dn}\hat{n}$

Theorem 3: Grad f is vector in the direction of which the maximum value of directional derivative of f ie. $\frac{df}{ds}$ occurs.

3.2 GRADIENT OF A SCALAR FIELD

Definition: Let $f(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then the gradient of f, written as ∇f or grad f is defined as

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f = \left(\Sigma \hat{i}\frac{\partial}{\partial x}\right) f$$

3.2.1 Properties of Gradient

Theorem 1. If f and g are two scalar point functions, then

grad $(f \pm g)$ = grad $f \pm$ grad g

or $\nabla(f \pm g) = \nabla f \pm \nabla g$

Proof.

$$\nabla(f \pm g) = \sum \hat{i}\frac{\partial}{\partial x}(f \pm g)$$

$$= \sum \hat{i}\frac{\partial}{\partial x} f \pm \sum \hat{i}\frac{\partial}{\partial x} g$$

$$= \left(\sum \hat{i}\frac{\partial}{\partial x}\right) f \pm \left(\sum \hat{i}\frac{\partial}{\partial x}\right) g$$

$$= \nabla f \pm \nabla g$$

Theorem 2. The necessary & sufficient condition for a scalar point function to be constant is that $\nabla f = 0$

Proof. If $f(x, y, z)$ is a constant

Then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$

So, $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = 0$

Hence, the condition is necessary

Conversely, if $\nabla f = 0$

$$\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}=0$$

So, $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}=0$

$\therefore$ f is independent of x, y, z

$\therefore$ f is a constant. Hence, condition is sufficient

Theorem 3. If f and g are two scalar point functions, then

$$\text{grad}\,(fg) = f\,\text{grad}\, g + g\,\text{grad}\, f$$

or $$\nabla(fg) = f\nabla g + g\nabla f$$

Proof:
$$\nabla(fg)=\sum\hat{i}\frac{\partial}{\partial x}(fg)$$
$$=\sum\hat{i}\left(f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\right)$$
$$=f\sum\hat{i}\frac{\partial g}{\partial x}+g\sum\hat{i}\frac{\partial f}{\partial x}$$
$$=f\nabla g+g\nabla f$$

Theorem 4. If f and g are two point functions then

$$\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f-f\nabla g}{g^2}$$

Proof:
$$\nabla\left(\frac{f}{g}\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right)$$
$$=\sum\hat{i}\left[\frac{g\frac{\partial f}{\partial x}-f\frac{\partial f}{\partial x}}{g^2}\right]$$
$$=\frac{1}{g^2}\left[g\Sigma\hat{i}\frac{\partial f}{\partial x}-f\Sigma\hat{i}\frac{\partial g}{\partial x}\right]$$
$$=\frac{g\nabla f-f\nabla g}{g^2}$$

3.2.2 Tangent Plane to a Level Surface

Let us find equation of the tangent plane and normal to the surface $f(x, y, z) = c$.

Let O be the origin. Let $\vec{r}$ is the position vector of point P on surface at which tengent plane is drawn. Let Q be the arbitrary point on the tangent plane

Let $\vec{R}$ is position vector of point Q. $\vec{R}-\vec{r}$ is a vector lying in a tangent plane.

Let $\hat{n}$ be a normal to the surface at point P and hence normal to tangent plane.

$\hat{n}$ is parallel to ∇f at point P

So, $(\vec{R}-\vec{r})\cdot\hat{n}=0$

Or, $(\vec{R}-\vec{r})\cdot\nabla f=0$ (Equation of tangent in vector form)

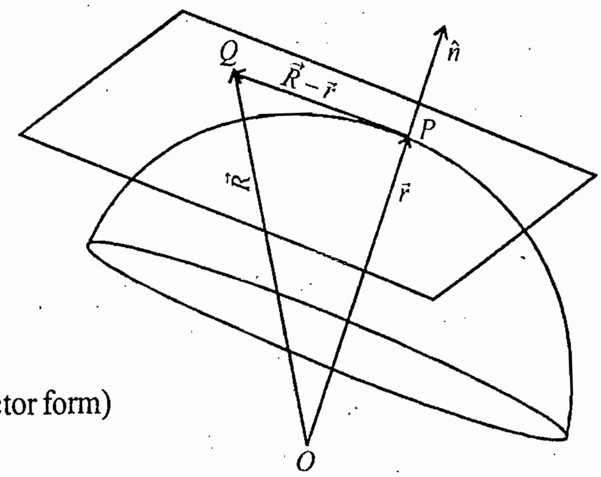


Figure 3.9

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$

So, in cartesian form, equation of tangent plane is given as

$$\left((X-x)\,\hat{i} + (Y-y)\hat{j} + (Z-z)\hat{k}\right) \cdot \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) = 0$$

$$(X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y} + (Z-z)\frac{\partial f}{\partial z} = 0$$

3.2.3 Equation of Normal to the level surface

Let O be the origin, let $\vec{r}$ is the position vector of point P on the surface at which normal is drawn. Let Q be the arbitrary point on the normal.

$(\vec{R} - \vec{r})$ is a vector lying on the normal and hence parallel to $\hat{n}$.

So, $(\vec{R} - \vec{r}) \times \hat{n} = 0$

or $(\vec{R} - \vec{r}) \times \nabla f = 0$

In cartesian form, Let $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

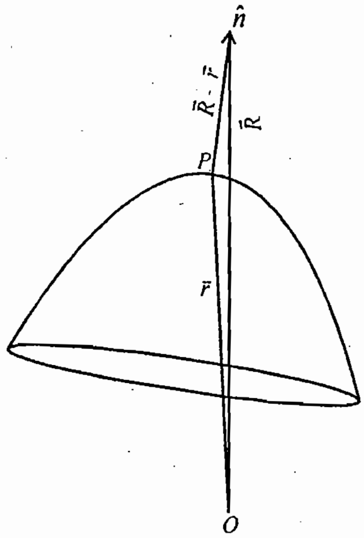


Figure 3.10

$$(\vec{R} - \vec{r}) \times \nabla f = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ X-x & Y-y & Z-z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = 0$$

$$\left((Y-y)\frac{\partial f}{\partial z} - (Z-z)\frac{\partial f}{\partial y}\right)\hat{i} + \left((Z-z)\frac{\partial f}{\partial x} - (X-x)\frac{\partial f}{\partial z}\right)\hat{j} + \left((X-x)\frac{\partial f}{\partial y} - (Y-y)\frac{\partial f}{\partial x}\right)\hat{k} = 0$$

or, $$\left((Y-y)\frac{\partial f}{\partial z} - (Z-z)\frac{\partial f}{\partial y}\right) = \left((Z-z)\frac{\partial f}{\partial x} - (X-x)\frac{\partial f}{\partial z}\right) = \left((X-x)\frac{\partial f}{\partial y} - (Y-y)\frac{\partial f}{\partial x}\right) = 0$$

or, $$\frac{X-x}{\frac{\partial f}{\partial x}} = \frac{Y-y}{\frac{\partial f}{\partial y}} = \frac{Z-z}{\frac{\partial f}{\partial z}}$$

SOLVED EXAMPLES (OBJECTIVE)

1. If $\hat{A} = x^2yz\,\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}$, $\vec{B} = 2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k}$, then value of $\dfrac{\partial^2}{\partial x \partial y}(\vec{A}\times\vec{B})$ at $(1,0,-2)$ is equal to

(a) 0 (b) $-4\hat{i} - 8\hat{j}$ (c) $6\hat{i} - 8\hat{j}$ (d) $-4\hat{i} - 4\hat{j}$

Ans. (b)

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= (2x^3z^3 - xyz^2)\,\hat{i} + (2xz^3 + x^4yz)\,\hat{j} + (x^2y^2z + 4xz^4)\,\hat{k}$$

$$\frac{\partial}{\partial y}(\vec{A}\times\vec{B}) = -xz^2\,\hat{i} + x^4z\,\hat{j} + 2x^2yz\,\hat{k}$$

$$\frac{\partial^2}{\partial x \partial y}(\vec{A}\times\vec{B}) = -z^2\,\hat{i} + 4x^3z\,\hat{j} + 4xyz\,\hat{k}$$

So, at $(1, 0, -2)$, $\dfrac{\partial^2}{\partial x \partial y}(\vec{A}\times\vec{B}) = -4\hat{i} - 8\hat{j}$

2. If $f(x, y, z) = 3x^2y - y^3z^2$, then grad f at the point $(1, -2, -1)$ is equal to

(a) $-12\hat{i} - 9\hat{j} - 16\hat{k}$ (b) $-6\hat{i} - 4\hat{j} - 2\hat{k}$ (c) $\hat{i} - 2\hat{j} - 4\hat{k}$ (d) 0

Ans. (a)

$$f = 3x^2y - y^3z^2$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\,\hat{j} - 2y^3z\,\hat{k}$$

At $(1, -2, -1)$, $\nabla f = -12\hat{i} - 9\hat{j} - 16\hat{k}$

3. The gradient of $f(r)$, is equal to

(a) $\dfrac{f'(r)}{r}\vec{r}$ (b) $f'(r)\dfrac{\vec{r}}{r^2}$ (c) 0 (d) $f'(r)\vec{r}$

Ans. (a)

$$\nabla f(r) = \sum \hat{i}\frac{\partial}{\partial x}(f(r))$$

$$= \sum \hat{i} f'(r)\frac{\partial r}{\partial x}$$

$$= \sum \hat{i} f'(r)\frac{x}{r}$$

$$= \frac{f'(r)}{r}\sum \hat{i}x$$

$$= \frac{f'(r)}{r}\vec{r}$$

4. $\nabla f(r)\times\vec{r}$ is equal to

(a) 0 (b) $\dfrac{\vec{r}}{r}$ (c) $\dfrac{\vec{r}}{r^2}$ (d) $\dfrac{\vec{r}}{r^3}$

Ans. (a)

$$\nabla f(r) = \frac{f'(r)}{r}\vec{r} \text{ (as solved in previous question)}$$

$$\nabla f(r) \times \vec{r} = 0$$

5. $\nabla\left(\frac{1}{r}\right)$ is equal to

(a) $-\frac{\vec{r}}{r^3}$ (b) 0 (c) $-\frac{\vec{r}}{r^2}$ (d) $\frac{\vec{r}}{r}$

Ans. (a)

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= \sum \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) \\ &= \sum \hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} \\ &= \sum \hat{i}\left(-\frac{1}{r^2}\right)\frac{x}{r} \\ &= -\frac{1}{r^3}\sum \hat{i}x \\ &= -\frac{\vec{r}}{r^3}\end{aligned}$$

6. $\nabla \log r$ is equal to

(a) $\frac{\vec{r}}{r}$ (b) $\frac{\vec{r}}{r^2}$ (c) $\frac{\vec{r}}{r^3}$ (d) 0

Ans. (b)

$$\begin{aligned}\nabla \log r &= \sum \hat{i}\frac{\partial}{\partial x}\log r \\ &= \sum \hat{i}\frac{1}{r}.\frac{\partial r}{\partial x} \\ &= \frac{1}{r^2}\sum \hat{i}x = \frac{\vec{r}}{r^2}.\end{aligned}$$

7. ∇r^n is equal to

(a) $(n-1)r^{n-1}\vec{r}$ (b) $nr^{n-1}\vec{r}$ (c) $nr^{n-2}\vec{r}$ (d) $(n-2)r^{n-1}\vec{r}$

Ans. (c)

$$\begin{aligned}\nabla r^n &= \sum \hat{i}\frac{\partial}{\partial x}r^n \\ &= \sum \hat{i}\, nr^{n-1}\frac{\partial r}{\partial x} \\ &= nr^{n-2}\sum \hat{i}x \\ &= nr^{n-2}\vec{r}\end{aligned}$$

8. If $\vec{a}$ is constant vector & $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then grad $(\vec{r}\cdot\vec{a})$ is equal to

(a) $\vec{a}$ (b) 0 (c) $(\vec{a}\cdot\vec{r})\vec{r}$ (d) $2\vec{a}$

Ans. (a)

$$\begin{aligned}\nabla(\vec{r}\cdot\vec{a}) &= \sum \hat{i}\frac{\partial}{\partial x}(\vec{r}\cdot\vec{a}) \\ &= \sum \hat{i}\left(\frac{\partial \vec{r}}{\partial x}\cdot\vec{a}\right)\end{aligned}$$

$$= \Sigma\hat{i}(\hat{i}\cdot\vec{a})$$
$$= \vec{a}$$

9. Let $\vec{a}\,\&\,\vec{b}$ are constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ grad$[\vec{r}\ \vec{a}\ \vec{b}]$ is equal to

(a) 0 (b) $\vec{a}\times\vec{b}$ (c) $\vec{b}\times\vec{a}$ (d) $\vec{r}\times(\vec{a}\times\vec{b})$

Ans. (b)

$$\text{grad}[\vec{r}\ \vec{a}\ \vec{b}] = \sum\hat{i}\frac{\partial}{\partial x}(\vec{r}\cdot(\vec{a}\times\vec{b}))$$
$$= \sum\hat{i}\left(\frac{\partial\vec{r}}{\partial x}\cdot(\vec{a}\times\vec{b})\right)$$
$$= \sum\hat{i}(\hat{i}\cdot(\vec{a}\times\vec{b}))$$
$$= \vec{a}\times\vec{b}$$

10. If $\vec{a}$ is a constant vector, ϕ is scalar field $(\vec{a}\cdot\nabla)\phi$ is equal to

(a) $\vec{a}$ (b) $\vec{a}\cdot\nabla\phi$ (c) 0 (d) ϕ

Ans. (b)

Let $$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{a}\cdot\nabla = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$
$$(\vec{a}\cdot\nabla)\phi = a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z}$$
$$= \vec{a}\cdot\nabla\phi$$

11. If $\vec{a}$ is constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $(\vec{a}\cdot\nabla)\vec{r}$ is equal to

(a) $\vec{a}$ (b) $(\vec{a}\cdot\vec{r})\vec{r}$ (c) $3\vec{a}$ (d) $2\vec{a}$

Ans. (a)

Let $$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{a}\cdot\nabla = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$
$$(\vec{a}\cdot\nabla)\vec{r} = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$$

12. The unit normal vector to the level surface $x^2 + y^2 - z = 4$ at point $(1, 1, -2)$ is

(a) $\frac{1}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{1}{3}\hat{k}$ (b) $\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{1}{3}\hat{k}$ (c) $\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$ (d) $\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$

Ans. (c)

Normal vector lies in direction of ∇f

So, $$\hat{n} = \frac{\nabla f}{|\nabla f|}$$
$$f = x^2 + y^2 - z$$
$$\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

At $(1, 1, -2)$,
$$\nabla f = 2\hat{i} + 2\hat{j} - \hat{k}$$
$$|\nabla f| = \sqrt{9} = 3$$

So, $$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$$
$$= \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

13. The directional derivative of $f(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of vector $2\hat{i} - \hat{j} - 2\hat{k}$ is

(a) $\frac{21}{3}$ (b) $\frac{12}{7}$ (c) $\frac{13}{3}$ (d) $\frac{37}{3}$

Ans. (d)

$$\nabla f = (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

At $(1, -2, -1)$, $\nabla f = 8\hat{i} - \hat{j} - 10\hat{k}$

So, directional derivative of f in direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is equal to

$$\nabla f.\hat{a} = \frac{1}{3}(8\hat{i} - \hat{j} - 10\hat{k}) \cdot (2\hat{i} - \hat{j} - 2\hat{k})$$
$$= \frac{37}{3}$$

14. The point P closest to origin on the plane $2x + y - z - 5 = 0$ is

(a) $\left(\frac{1}{3}, \frac{5}{6}, \frac{-7}{2}\right)$ (b) $\left(\frac{5}{6}, \frac{5}{3}, \frac{-5}{3}\right)$ (c) $\left(\frac{5}{3}, \frac{5}{6}, \frac{-5}{6}\right)$ (d) $\left(\frac{5}{3}, \frac{1}{6}, \frac{-5}{6}\right)$

Ans. (c)

Closest point will be foot of perpendicular from origin

$$S = 2x + y - z - 5 = 0$$
$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2\hat{i} + \hat{j} - \hat{k}}{\sqrt{6}}$$

Coordinate of $P = \left(\frac{2}{\sqrt{6}}r, \frac{1}{\sqrt{6}}r, \frac{-1}{\sqrt{6}}r\right)$

It lies on S

So, $$r = \frac{5}{\sqrt{6}}$$

Hence, $$P = \left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right)$$

15. The temperature T at a surface is given by $T = x^2 + y^2 - z$. In which direction a mosquito at the point $(4, 4, 2)$ on the surface will fly so that it cools fastest?

(a) $8\hat{i} + 8\hat{j} - \hat{k}$ (b) $-8\hat{i} - 8\hat{j} + \hat{k}$ (c) $\hat{i} - \hat{j} + 2\hat{k}$ (d) $\hat{i} + \hat{j} - \hat{k}$

Ans. (b)

$$T = x^2 + y^2 - z$$

Direction of fastest cooling will lie in direction opposite to the direction of gradient i.e. $-\nabla T$

$$\nabla T = 2x\hat{i} + 2y\hat{j} - \hat{k}$$
$$= 8\hat{i} + 8\hat{j} - \hat{k}$$

16. The scalar function f which corresponds to $\vec{V} = \nabla f$

where $$\vec{V} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$ is

(a) $\sqrt{x^2+y^2+z^2}+c$ (b) $\dfrac{1}{\sqrt{x^2+y^2+z^2}}+c$ (c) $(x^2+y^2+z^2)^{\frac{-3}{2}}+c$ (d) xyz

Ans. (a)

$$f=\sqrt{x^2+y^2+z^2}+c$$
$$\nabla f=\frac{\vec{r}}{r}$$

17. One of the point at which the derivative of the function $f(x,y)=x^2-xy-y+y^2$ vanishes along the direction $\dfrac{\hat{i}+\sqrt{3}\hat{j}}{2}$ is

(a) $\left(-1,\dfrac{2}{2\sqrt{3}+1}\right)$ (b) $\left(-1,\dfrac{2}{2\sqrt{3}-1}\right)$ (c) $\left(1,\dfrac{2}{2\sqrt{3}+1}\right)$ (d) $\left(1,\dfrac{2}{2\sqrt{3}-1}\right)$

Ans. (b)

$$\nabla f=(2x-y)\hat{i}-(x+1-2y)\hat{j}$$

Directional derivative in direction given by $\dfrac{\hat{i}+\sqrt{3}\hat{j}}{2}$

$$=\frac{1}{2}(2x-y)-\frac{\sqrt{3}}{2}(x+1-2y)$$
$$=\frac{2-\sqrt{3}}{2}x-\frac{(1-2\sqrt{3})}{2}y-\frac{\sqrt{3}}{2}$$

It becomes zero at $\left(-1,\dfrac{2}{2\sqrt{3}-1}\right)$

18. Which of the following is a unit normal vector to the surface $z=xy$ at $P(2,-1,-1)$?

(a) $\dfrac{\hat{i}-2\hat{j}+\hat{k}}{\sqrt{6}}$ (b) $\hat{i}-2\hat{j}+\hat{k}$ (c) $-\hat{i}+2\hat{j}+\hat{k}$ (d) $\dfrac{-\hat{i}+2\hat{j}+\hat{k}}{\sqrt{6}}$

Ans. (a)

The surface is $f=xy-z=0$

$$\nabla f=y\hat{i}+x\hat{j}-\hat{k}$$
$$=-\hat{i}+x\hat{j}-\hat{k}$$
$$\hat{n}=\frac{\nabla f}{|\nabla f|}=\frac{\hat{i}-2\hat{j}+\hat{k}}{\sqrt{6}}$$

19. Let $f(x,y)=\ln\sqrt{x+y}$ and $g(x,y)=\sqrt{x+y}$. Then the value of $\nabla^2(fg)$ at $(1,0)$

(a) $-\dfrac{1}{2}$ (b) 0 (c) $\dfrac{1}{2}$ (d) 1

Ans. (b)

$$f=\ln(x+y)^{1/2},\ g=\sqrt{x+y}$$
$$fg=\sqrt{x+y}\ln\sqrt{x+y}$$
$$\nabla^2 fg=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)fg$$

$$\frac{\partial}{\partial x}(fg)=\frac{1}{2\sqrt{x+y}}\ln\sqrt{x+y}+\sqrt{x+y}\frac{1}{\sqrt{x+y}}\cdot\frac{1}{2\sqrt{x+y}}$$

$$\frac{\partial^2 fg}{\partial x^2}=-\frac{1}{4}(x+y)^{-3/2}\ln(x+y)+\frac{1}{2(x+y)}\cdot\frac{1}{2\sqrt{x+y}}-\frac{1}{4}(x+y)^{-3/2}$$

$$\nabla^2 fg=0$$

20. The spheres

$x^2+y^2+z^2=1$ and $x^2+(y-\sqrt{3})^2+z^2=4$ intersect at an angle

(a) 0 (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

Ans. (d)

$$x^2+y^2+z^2=1$$

$$x^2+y^2+z^2-2\sqrt{3}y=1$$

They intersect at plane $y=0$

$(0,0,1)$ is one point of intersection which is lying on the both sphere

Let us find normal vector at this point and find angle between them

$$\hat{n}_1=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\hat{n}_2=\frac{x\hat{i}+(y-\sqrt{3})\hat{j}+z\hat{k}}{2}$$

$$\cos\theta=\hat{n}_1.\hat{n}_2=\frac{1}{2} \text{ at point } (0,0,1)$$

$$\theta=\pi/3$$

21. Let $\theta,\ 0\le\theta\le\pi$ be the angle between the planes

$x-y+z=0$ and $2x-z=4$

The value of θ is

(a) $\cos^{-1}\left(\frac{1}{5}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$ (c) $\cos^{-1}\left(\frac{1}{\sqrt{15}}\right)$ (d) $\cos^{-1}\left(\frac{3}{\sqrt{15}}\right)$

Ans. (c)

$$x-y+z=3 \Rightarrow x-y+z-3=0$$

$$2x-z=4 \Rightarrow 2x-z-4=0$$

Let us find angle between their normal

$$\hat{n}=\frac{\nabla f}{|\nabla f|}$$

$$\cos\theta=\hat{n}_1.\hat{n}_2=\frac{1}{\sqrt{15}}$$

$$\hat{n}_2=\frac{2\hat{i}-\hat{k}}{\sqrt{5}}$$

$$\hat{n}_1=\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\Rightarrow \qquad \theta=\cos^{-1}\frac{1}{\sqrt{15}}$$

22. $f(x, y) = xy^2 + yx^2$

Suppose the directional derivative of f in the direction of the unit vector (u_1, u_2) at the point $(1, -1)$ is 1. Then among the following (u_1, u_2) is

(a) $(-1, 0)$ (b) $(0, 1)$ (c) $(1, 0)$ (d) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Ans. (a)

$$f = xy^2 + yx^2$$

$$\nabla f = (y^2 + 2xy)\hat{i} + (x^2 + 2xy)\hat{j}$$

At $(1, -1)$ $\nabla f = -\hat{i} - \hat{j}$

$$\hat{n} = u_1\hat{i} + u_2\hat{j}$$

Directional derivative of f in direction of unit vector (u_1, u_2) is

$$\nabla f \cdot \hat{n} = 1$$

$$\Rightarrow \quad -u_1 - u_2 = 1$$

$u_1 = -1, u_2 = 0$ satisfies above equation

23. For what values of a and b, the directional derivative of $u(x, y, z) = ax^2yz + bxy^2z$ at $(1, 1, 1)$ along $\hat{i} + \hat{j} - 2\hat{k}$ is $\sqrt{6}$ and along $\hat{i} - \hat{j} + 2\hat{k}$ is $3\sqrt{6}$?

(a) $a = 6, b = 0$ (b) $a = 0, b = 6$ (c) $a = 6, b = 6$ (d) $a = -6, b = 6$

Ans. (a)

$$\nabla u = (2axyz + by^2z)\hat{i} + (ax^2z + 2bxyz)\hat{j} + (ax^2y + bxy^2)\hat{k}$$

The directional derivative of $u(x, y, z)$ along $(\hat{i} + \hat{j} - 2\hat{k})$ at $(1, 1, 1)$

$$(2a+b)\hat{i} + (a+2b)\hat{j} + (a+b)\hat{k}.\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}(2a+b+a+2b-2a-2b)$$

$$= \frac{a+b}{\sqrt{6}} = \sqrt{6} \text{ (Given)}$$

So, $a + b = 6$ ... (1)

The directional derivative of $u(x, y, z)$ along $(\hat{i} - \hat{j} + 2\hat{k})$ at $(1, 1, 1)$

$$= ((2a+b)\hat{i} + (a+2b)\hat{j} + (a+b)\hat{k}) \cdot \frac{\hat{i}-\hat{j}+2\hat{k}}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}(3a+b) = 3\sqrt{6} \qquad \text{(Given)}$$

$3a + b = 18$ ... (2)

Solving (1) & (2)

$$a = 6, b = 0$$

24. If $f(x, y, z) = x - y$ and $\nabla\left(\frac{f}{g}\right) = \frac{1}{z}(\hat{i} - \hat{j}) - \left(\frac{x-y}{z^2}\right)\hat{k}$ then $g(x, y, z)$ is

(a) xyz (b) x (c) y (d) z

Ans. (d)

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

$$= \frac{\nabla f}{g} - \frac{f}{g^2}\nabla g$$

Given $\nabla\left(\frac{f}{g}\right)=\frac{1}{z}(\hat{i}-\hat{j})-\left(\frac{x-y}{z^2}\right)\hat{k}$

On comparing $g(x,y,z)=z$

25. The directional derivative of $f(x,y,z)=z^2e^{\cos xy}$ at $\left(0,\frac{\pi}{2},1\right)$ along $(2\hat{i}-\hat{j}+2\hat{k})$ is

(a) $\frac{e}{3}$ (b) $\frac{3e}{4}$ (c) $\frac{4e}{3}$ (d) $\frac{e}{4}$

Ans. (c)

$$\nabla f=-yz^2e^{\cos xy}.\sin xy\,\hat{i}-xz^2e^{\cos xy}.\sin xy\,\hat{j}+2ze^{\cos xy}\hat{k}$$

Unit vector along $2\hat{i}-\hat{j}+2\hat{k}$ is given by

$$\hat{n}=\frac{1}{3}(2\hat{i}-\hat{j}+2\hat{k})$$

Directional derivative along $(2\hat{i}-\hat{j}+2\hat{k})$

$$\frac{df}{ds}=\nabla f\cdot\hat{n}=(-z^2e^{\cos xy}y\sin xy\,\hat{i}-z^2e^{\cos xy}x\sin xy\,\hat{j}+2ze^{\cos xy}\hat{k})\cdot\frac{(2\hat{i}-\hat{j}+2\hat{k})}{3}$$

$$=\frac{1}{3}(-2z^2ye^{\cos xy}\sin xy-xz^2e^{\cos xy}\sin xy+4ze^{\cos xy})$$

At $\left(0,\frac{\pi}{2},1\right)$, directional derivative

$$=\frac{4}{3}e$$

26. Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$. Then $\nabla\left(\vec{r}\cdot\nabla(\vec{r}\cdot\vec{r})\right)$ is equal to

(a) $2\vec{r}$ (b) $4\vec{r}$ (c) $\frac{\vec{r}}{2}$ (d) $\frac{\vec{r}}{4}$

Ans. (b)

$$\nabla(\vec{r}.\vec{r})=\sum\hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)=2\vec{r}$$

$$\vec{r}\cdot\nabla(\vec{r}.\vec{r})=2\vec{r}.\vec{r}=2r^2$$

$$\nabla\left(\vec{r}\cdot\nabla(\vec{r}\cdot\vec{r})\right)=2\nabla(r^2)$$

$$=4\vec{r}$$

27. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then $\nabla|\vec{r}|^4$ equals

(a) $4|\vec{r}|$ (b) $4|\vec{r}|^2\vec{r}$ (c) $4|\vec{r}|\vec{r}$ (d) $4|\vec{r}|^3$

Ans. (b)

$$\nabla(|r|^4)=\nabla(r^4)$$

$$=\Sigma\hat{i}\frac{\partial}{\partial x}(r^4)$$

$$=4r^3\frac{\Sigma\hat{i}\,x}{r}$$

$$=4r^2\vec{r}$$

28. Let $T(x,y,z)=xy^2+2z-x^2z^2$ be the temperature at the point (x,y,z). The unit vector in the direction in which the temperature decreases most rapidly at $(1,0,-1)$ is

(a) $-\frac{1}{\sqrt{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}$ (b) $\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{k}$

(c) $\frac{2}{\sqrt{14}}\hat{i}+\frac{3}{\sqrt{14}}\hat{j}+\frac{1}{\sqrt{14}}\hat{k}$ (d) $-\left(\frac{2}{\sqrt{14}}\hat{i}+\frac{3}{\sqrt{14}}\hat{j}+\frac{1}{\sqrt{14}}\hat{k}\right)$

Ans. (b)

Temperature increases most rapidly in the direction of ∇T.

$\nabla T=(y^2-2xz^2)\hat{i}+2xy\hat{j}+(2-2x^2z)\hat{k}$

At $(1,0,-1), \nabla T=-2\hat{i}+4\hat{k}$

Unit vector in the direction of $\nabla T=-\frac{1}{\sqrt{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}$

So, temperature decreases most rapidly in the direction of $-\nabla T$.

i.e., $\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{k}$

29. The equation of a surface of revolution is

$$z=\pm\sqrt{\frac{3}{2}x^2+\frac{3}{2}y^2}$$

The unit normal to the surface at the point

$$A\left(\sqrt{\frac{2}{3}},0,1\right) \text{ is}$$

(a) $\sqrt{\frac{3}{5}}\hat{i}+\frac{2}{10}\hat{k}$ (b) $\sqrt{\frac{3}{5}}\hat{i}-\frac{2}{\sqrt{10}}\hat{k}$ (c) $\sqrt{\frac{3}{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}$ (d) $\sqrt{\frac{3}{10}}\hat{i}+\frac{2}{\sqrt{10}}\hat{k}$

Ans. (b)

Equation of surface is

$$z^2=\frac{3}{2}(x^2+y^2)$$

$$F=3x^2+3y^2-2z^2=0$$

Unit normal to the surface

$$\hat{n}=\frac{\nabla F}{|\nabla F|}=\frac{6x\hat{i}+6y\hat{j}-4z\hat{k}}{2\sqrt{9x^2+9y^2+4z^2}}$$

$$=\frac{3x\hat{i}+3y\hat{j}-2z\hat{k}}{\sqrt{9x^2+9y^2+4z^2}}$$

$$=\frac{3\sqrt{\frac{2}{3}}\hat{i}-2\hat{k}}{\sqrt{9\times\frac{2}{3}+4}}=\frac{\sqrt{6}\,\hat{i}-2\hat{k}}{\sqrt{10}}$$

$$=\sqrt{\frac{3}{5}}\hat{i}-\frac{2}{\sqrt{10}}\hat{k}$$

SOLVED EXAMPLES (SUBJECTIVE)

1. Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of line PQ where Q is the point $(5, 0, 4)$.

Solution.

Here, function $f = x^2 - y^2 + 2z^2$

$$\nabla f = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

At $(1, 2, 3)$, $\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

Now, vector $\overrightarrow{PQ}$ = position vector of Q – position vector of P

$$= (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= 4\hat{i} - 2\hat{j} + \hat{k}$$

Unit vector in direction of $\overrightarrow{PQ}$,

$$\hat{a} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

So, directional derivative of f in the direction of $\hat{a} = \nabla f \cdot \hat{a}$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

$$= \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}$$

2. What is the greatest rate of increase of $u = xyz^2$ at the point $(1, 0, 3)$?

Solution.

$$u = xyz^2$$
$$\nabla u = yz^2\hat{i} + xz^2\hat{j} + 2xyz\hat{k}$$

At $(1, 0, 3)$, $\nabla u = 9\hat{j}$

The greatest rate of increase of f lie in the direction of ∇f.

So, maximum value of directional derivative

$= \nabla u \cdot \hat{a}$ with $\hat{a}$ being unit vector parallel to ∇u

$$= |\nabla u| = 9$$

3. Find the directional derivative of

(i) $4xz^3 - 3x^2y^2z^2$ at $(2, -1, 2)$ along z axis.

(ii) $x^2yz + 4xz^2$ at $(1, -2, 1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$.

Solution.

(i) $f = 4xz^3 - 3x^2y^2z^2$

$$\nabla f = (4z^3 - 6xy^2z^2)\hat{i} - 6x^2yz^2\hat{j} + (12xz^2 - 6x^2y^2z)\hat{k}$$

At $(2, -1, 2)$, $\nabla f = -16\hat{i} + 96\hat{j} + 48\hat{k}$

Along z axis, the directional derivative along z axis

$$= \nabla f \cdot \hat{k} = 48$$

(ii) $f = x^2yz + 4xz^2$

$$\nabla f = (2xyz + 4z^2)\hat{i} + x^2z\hat{j} + (x^2y + 8zx)\hat{k}$$

At $(1, -2, 1)$, $\nabla f = \hat{j} + 6\hat{k}$

Unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$

$$\hat{a} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{9}} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{3}$$

Directional derivative in direction of $2\hat{i}-\hat{j}-2\hat{k}$

$$= \nabla f \cdot \hat{a} = (\hat{j}+6\hat{k})\cdot\left(\frac{2\hat{i}-\hat{j}-2\hat{k}}{3}\right) = -\frac{13}{3}$$

4. Find the directional derivative of $f(x, y) = x^2y^3 + xy$ at the point (2, 1) in the directional of a unit vector which makes an angle of $\pi/3$ with x axis.

Solution.

$$f(x, y) = x^2y^3 + xy$$

$$\nabla f = (2xy^3 + y)\hat{i} + (3x^2y^2 + x)\hat{j}$$

At (2, 1), $\nabla f = 5\hat{i} + 14\hat{j}$

Unit vector making an angle of $\pi/3$ with x axis

$$\hat{a} = \cos\frac{\pi}{3}\hat{i} + \sin\frac{\pi}{3}\hat{j}$$

$$= \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{3}\hat{j}.$$

So, directional derivative of f in the direction of unit vector making angle of $\frac{\pi}{3}$ with the x axis

$$= \nabla f \cdot \hat{a}$$

$$= (5\hat{i}+14\hat{j})\cdot\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right)$$

$$= \frac{5+14\sqrt{3}}{2}$$

5. Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, –1, 2)

Solution.

The two surface are orthogonal at point P if the respective normal to the surface are perpendicular to each other.

The surface S_1 is given by

$$S_1 : ax^2 - byz - (a+2)x = 0$$

Gradient of S_1, $\nabla S_1 = (2ax - (a+2))\hat{i} - bz\,\hat{j} - by\,\hat{k}$

At $(1, -1, 2)$ $\nabla S_1 = (a-2)\hat{i} - 2b\,\hat{j} - b\hat{k}$

Unit normal vector to S_1

$$\hat{n}_1 = \frac{\nabla S_1}{|\nabla S_1|}$$

The surface S_2 is given by

$$S_2 : 4x^2y + z^3 = 4$$

Gradient of S_2, $\nabla S_2 = 8xy\,\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

At $(1, -1, 2)$ $\nabla S_2 = 8\hat{i} + 4\hat{j} + 12\hat{k}$

Normal to S_2, $\hat{n}_2 = \frac{\nabla S_2}{|\nabla S_2|}.$

Two surface S_1 & S_2 are orthogonal

So, $\hat{n}_1.\hat{n}_2 = 0$

$$\frac{\nabla S_1}{|\nabla S_1|}.\frac{\nabla S_2}{|\nabla S_2|} = 0$$

$$\nabla S_1.\nabla S_2 = 0$$

$\Rightarrow$ $((a-2)\hat{i} - 2b\,\hat{j} + b\hat{k}).(-8\hat{i} + 4\hat{j} + 12\hat{k}) = 0$

$\Rightarrow$ $-8(a-2) - 8b + 12b = 0$

$\Rightarrow$ $-8a + 4b = -16$

Point $(1,-1,2)$ lies on S_1

So, $a + 2b = a + 2$

$\Rightarrow$ $b = 1$

So, $a = \frac{5}{2}$

6. Find the values of constant a, b & c so that the directional derivative of the function. $f = axy^2 + byz + cz^2x^3$ at the point $(1, 2, -1)$ has maximum magnitude 64 in the direction of parallel to $z-$axis

Solution.

The function $f = axy^2 + byz + cz^2x^3$

$$\nabla f = \sum \hat{i}\frac{\partial}{\partial x} f = (ay^2 + 3cz^2x^2)\,\hat{i} + (2axy + bz)\,\hat{j} + (by + 2czx^3)\hat{k}$$

At $(1,2,-1)$ $\nabla f = (4a+3c)\hat{i} + (4a-b)\hat{j} + (2b-2c)\hat{k}$

The directional derivative of f is maximum along ∇f and it is given that maximum value of directional derivative is along z axis. So, ∇f is parallel to z axis. So, its x & y component should be zero.

So, $4a + 3c = 0 \Rightarrow c = -\frac{4a}{3}$

$4a - b = 0 \Rightarrow b = 4a$

So, $\nabla f = (2b - 2c)\hat{k}$

$$= \frac{32}{3}a\hat{k}$$

Maximum value of directional derivative is equal to $|\nabla f| = 64$

$\Rightarrow$ $\frac{32a}{3} = 64$

$a = 6$

So, $b = 24$

$c = -8$

7. Find the directional derivative of $f = x^2yz^3$ along $x = e^{-t}$, $y = 1 + 2\sin t$, $z = t - \cos t$ at $t = 0$

Solution.

The function

$$f = x^2yz^3$$

$$\nabla f = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$$

For $t = 0$, $x = e^{-t} = 1$

$y = 1 + 2\sin t = 1$

$z = t - \cos t = -1$

So, at $(1,1-1)$ $\nabla f = -2\hat{i} - \hat{j} + 3\hat{k}$

The curve is described by vector

$$\vec{r} = e^{-t}\hat{i} + (1+2\sin t)\hat{j} + (t-\cos t)\hat{k}$$

$$\vec{t} = \frac{d\vec{r}}{dt} = -e^{-t}\hat{i} + 2\cos t\,\hat{j} + (1+\sin t)\hat{k}$$

At $t=0$ $\quad \vec{t} = -\hat{i} + 2\hat{j} + \hat{k}$

Unit vector along tangent,

$$\hat{t} = \frac{-\hat{i}+2\hat{j}+\hat{k}}{\sqrt{6}}$$

Directional derivative along the curve at $t=0$

$$= \nabla f.\hat{t}$$

$$= (-2\hat{i} - \hat{j} + 3\hat{k}).\frac{(-\hat{i}+2\hat{j}+\hat{k})}{\sqrt{6}}$$

$$= \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

8. If $\vec{r}_1$ and $\vec{r}_2$ are the vector joing the fixed point. $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ respectively to a variable point $P(x, y, z)$ then find the values of grad $(\vec{r}_1.\vec{r}_2)$ & $(\vec{r}_1 \times \vec{r}_2)$.

Solution.

The vector

$$A\vec{P} = \vec{r}_1 = \text{position vector of } P - \text{position vector of } A$$

$$= (x\hat{i} + y\hat{j} + 2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= (x-x_1)\hat{i} + (y-y_1)\hat{j} + (z-z_1)\hat{k}$$

Similary, $\quad B\vec{P} = \vec{r}_2 = (x-x_2)\hat{i} + (y-y_2)\hat{j} + (z-z_2)\hat{k}$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \end{vmatrix}$$

$$= [(y-y_1)(z-z_2)-(y-y_2)(z-z_1)]\hat{i}$$

$$+[(x-x_2)(z-z_1)-(x-x_1)(z-z_2)]\hat{j}$$

$$+[(y-y_1)(z-z_2)-(y-y_2)(z-z_1)]\hat{k}$$

$$= \left[y(z_1-z_2) + z(y_2-y_1) + (y_1z_2 - y_2z_1)\right]\hat{i}$$

$$+\left[z(x_1-x_2) + x(z_2-z_1) + (z_1x_2 - z_2x_1)\right]\hat{j}$$

$$+\left[x(y_1-y_2) + y(x_2-x_1) + (x_1y_2 - x_2y_1)\right]\hat{k}$$

$$\vec{r}_1.\vec{r}_2 = (x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2)$$

$$\nabla(\vec{r}_1.\vec{r}_2) = \sum \hat{i}\frac{\partial}{\partial x}(\vec{r}_1.\vec{r}_2)$$

$$= \sum \hat{i}\,(2x - x_1 - x_2)$$

$$= \sum \hat{i}\left[(x-x_1)+(x-x_2)\right]$$

$$= \sum \hat{i}(x-x_1) + \sum \hat{i}(x-x_2)$$

$$= \vec{r}_1 + \vec{r}_2$$

9. Find the equation of tangent plane and normal to the surface $2xz^2 - 3xy + 4x = 1$ at the point (1,1,2)

Solution.

Equation of the surface is

$$f(x, y, z) = 2xz^2 - 3xy + 4x = 1$$

$$\nabla f = \Sigma \hat{i} \frac{\partial}{\partial x} f$$

$$= (2z^2 - 3y + 4)\hat{i} - 3x\hat{j} + 4xz\hat{k}$$

At (1,1,2) $\quad \nabla f = 9\hat{i} - 3\hat{j} + 8\hat{k}$

let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ is a position vector of any arbitrary point (x,y,z) on the tangent plane at point P.

The position vector of point P is $\vec{r} = \hat{i} + \hat{j} + 2\hat{k}$

Equation of tangent plane at point P is

$$(\vec{R} - \vec{r}) \cdot \text{grad } f = 0$$

$$\Rightarrow \quad (x-1)\frac{\partial f}{\partial x} + (y-1)\frac{\partial f}{\partial y} + (z-2)\frac{\partial f}{\partial z} = 0$$

$$\Rightarrow \quad 9(x-1) - 3(y-1) + 8(z-2) = 0$$

$$9x - 3y + 8z = 22$$

Equation of normal to the surface at point (1,1,2) is

$$\frac{x-1}{\frac{\partial f}{\partial x}} = \frac{y-1}{\frac{\partial f}{\partial y}} = \frac{z-2}{\frac{\partial f}{\partial z}}$$

$$\frac{x-1}{9} = \frac{y-1}{-3} = \frac{z-2}{8}$$

10. Find the equation of the tangent plane and normal to the surface $xyz = 2$ at the point (1,2,1).

Solution.

The equation of surface is

$$f(x, y, z) = xyz - 2 = 0$$

$$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

At point (1,2,1) $\quad \nabla f = 2\hat{i} + \hat{j} + 2\hat{k}$

Let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of an arbitrary point (x, y, z) on the tangent plane.

Position vector of point of contact (1, 2, 1)

$$\vec{r} = \hat{i} + 2\hat{j} + \hat{k}$$

Equation of tangent plane is

$$(\vec{R} - \vec{r}).\nabla f = 0$$

$$\Rightarrow \quad (x-1)\frac{\partial f}{\partial x} + (y-2)\frac{\partial f}{\partial y} + (z-1)\frac{\partial f}{\partial z} = 0$$

$$\Rightarrow \quad 2(x-1) + (y-2) + 2(z-1) = 6$$

$$2x + y + 2z = 6$$

Equation of normal to the surface at point (1,2,1)

$$\frac{x-1}{\frac{\partial f}{\partial x}} = \frac{y-2}{\frac{\partial f}{\partial y}} = \frac{z-1}{\frac{\partial f}{\partial z}}$$

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-1}{2}$$

11. Give the curve $x^2+y^2+z^2=1, x+y+z=1$ (intersection of two surfaces) find the equation of the tangent line at the point (1,0,0).

Solution.

Given is the curve of intersection of two surfaces

$$S_1 \equiv x^2+y^2+z^2=0 \quad S_2 \equiv x+y+z=1$$

$$\nabla S_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}, \nabla S_2 = \hat{i}+\hat{j}+\hat{k}$$

At point (1,0,0) $\nabla S_1 = 2\hat{i}$

$$\nabla S_2 = \hat{i}+\hat{j}+\hat{k}$$

The normal vector to surface S_1 & S_2 are given by

$$\hat{n}_1 = \frac{\nabla S_1}{|\nabla S_1|} = \hat{i}$$

$$\hat{n}_2 = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

Tangent to the curve of intersection will be perpendicular to both $\hat{n}_1$ & $\hat{n}_2$ ie. it lies in the direction of

$\hat{n}_1 \times \hat{n}_2$ ie. $\hat{i} \times \frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}$

$$= -\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$$

So, equation of tangent passing through (1,0,0) & parallel to vector $-\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$ is

$$\frac{x-1}{0} = \frac{y-0}{-1/\sqrt{3}} = \frac{z-0}{1/\sqrt{3}}$$

$$x=1, y+z=0$$

12. Find the angle between the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Solution.

Angle between two surfaces at a point will be equal to the angle between their respective normal at the point. So, let us find the unit normal vector to the surfaces at given point ie. $(2,-1,2)$

The given surfaces are

$$S_1 \equiv x^2+y^2+z^2=9$$

$$S_2 \equiv x^2+y^2-z=3$$

Their gradients are

$$\nabla S_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\nabla S_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$$

At $(2,-1,2)$

$$\nabla S_1 = 4\hat{i}-2\hat{j}+4\hat{k}$$

$$\nabla S_2 = 4\hat{i}-2\hat{j}-\hat{k}$$

The unit normal vectors to the surface S_1 & S_2 at point $(2,-1,2)$ are

$$\hat{n}_1 = \frac{\nabla S_1}{|\nabla S_1|} = \frac{4\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{36}} = \frac{2}{3}\hat{i}-\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}$$

$$\hat{n}_2 = \frac{\nabla S_2}{|\nabla S_2|} = \frac{4\hat{i}-2\hat{j}-\hat{k}}{\sqrt{21}} = \frac{4}{\sqrt{21}}\hat{i}-\frac{2}{\sqrt{21}}\hat{j}-\frac{1}{\sqrt{21}}\hat{k}$$

If θ is the angle between two vectors

then $\qquad \hat{n}_1.\hat{n}_2 = |\hat{n}_1||\hat{n}_2|\cos\theta$

$$\Rightarrow \qquad \cos\theta = \frac{8}{3\sqrt{21}} + \frac{2}{3\sqrt{21}} - \frac{2}{3\sqrt{21}}$$

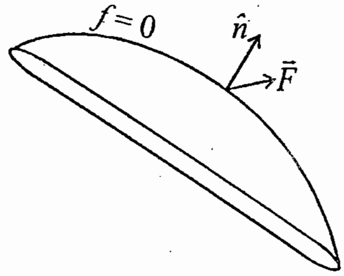
$$= \frac{8}{3\sqrt{21}}$$

So, $\qquad \theta = \cos^{-1}\frac{8}{3\sqrt{21}}$

13. If $\vec{F}$ & f are two point function, show that the components of the former tangential and normal to the level surface $f=0$ are $\dfrac{\nabla f \times (\vec{F} \times \nabla f)}{(\nabla f)^2}$ and $\dfrac{(\vec{F}.\nabla f)\nabla f}{(\nabla f)^2}$

Solution.

Unit normal vector to the surface $f=0$



$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

Magnitude of component of $\vec{F}$ normal to the surface $f=0$ is

$$\vec{F}.\hat{n} = \vec{F}.\frac{\nabla f}{|\nabla f|}$$

Component of $\vec{F}$ normal to the surface $f=0$

$$(\vec{F}.\hat{n})\hat{n} = \left(\vec{F}.\frac{\nabla f}{|\nabla f|}\right)\frac{\nabla f}{|\nabla f|}$$

$$= \frac{(\vec{F}.\nabla f)\nabla f}{(\nabla f)^2}$$

Component of $\vec{F}$ tangential to surface $f=0$

$$= \vec{F} - \text{normal component of } \vec{F}$$

$$= \vec{F} - \frac{(\vec{F}.\nabla f)\nabla f}{(\nabla f)^2}$$

$$= \frac{(\nabla f.\nabla f)\vec{F} - (\vec{F}.\nabla f)\nabla f}{(\nabla f)^2}$$

$$= \frac{\nabla f \times (\vec{F} \times \nabla f)}{(\nabla f)^2}$$

EXERCISE – 1

1. Find the unit normal to the surface $z = x^2 + y^2$ at the point $(-1, -2, 5)$.
2. Find the unit normal to the surface $x^4 - 3xyz + z^2 + 1 = 0$ at the point $(1,1,1)$.
3. Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.
4. Find the directional directive of the function $f = xy + yz + zx$ in the direction of the vector $2\hat{i} + 3\hat{j} + 6\hat{k}$ at the point $(3,1,2)$.
5. Find the maximum values of directional derivatives of $\phi = x^2yz$ at the point $(1, 4, 1)$.
6. Find the equations of the tangent plane and normal to the surface $x^2 + y^2 + z^2 = 9$ at the point $(2, 2, 1)$.

7. Find the equation of tangent plane and normal to the surface $z = x^2y^2$ at the point (2, 1, 5).

8. Find the values of constants a, b and c such that the maximum value of directional derivative of $f = axy^2 + byz + cx^2z^2$ at (1, –1, 1) is in direction parallel to y axis and has magnitude 6.

9. In what direction from the point (–1, 1, 1) is the directional derivative of $f = x^2yz^3$ a maximum? Compute its magnitude.

10. Find a unit normal to the surface $xy^2 + 2xz = 4$ at the point (2, –2, 3).

11. In what direction from the point (1, 3, 2) is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of maximum ?

12. Find the direction derivative of $\phi = 4xz^3 - 3x^2y^2z$ at (2, –1, 2) in the direction of $2\hat{i} - 3\hat{j} + 6\hat{k}$.

13. Prove that the directional derivative of the scalar function $\phi(x, y, z) = (x^2 + y^2 + z^2)\, e^{-\sqrt{x^2+y^2+z^2}}$ vanishes at the origin (0,0,0) in the direction of normal to the surface $\phi(x, y, z) = \text{constant}$.

ANSWERS

1. $\dfrac{1}{\sqrt{21}}(2\hat{i} + 4\hat{j} + \hat{k})$ **2.** $\dfrac{1}{\sqrt{11}}(\hat{i} - 3\hat{j} - \hat{k})$ **3.** –13/3 **4.** 45/7

5. 9 **10.** $\dfrac{1}{3}(-\hat{i} + 2\hat{j} + 2k)$ **11.** $2\sqrt{14}$ **12.** $\dfrac{376}{7}$

Bhaskara

Bhaskara (600 AD – 680 AD) Commonly called Bhaskara I to avoid confusion with the 12th century mathematician Bhaskara II) was a 7th century Indian mathematician, who was apparently the first to write numbers in the Hindu-Arabic decimal system with a circle for the zero, and who gave a unique and remarkable rational approximation of the sine function in his commentary on Aryabhata's work. This commentary, written in 629 AD, is the oldest known prose work in Sanskrit on mathematics and astronomy. He also wrote two astronomical works in the line of Aryabhata's school, the Mahabhaskariya and the Laghubhaskariya.

Little is known about Bhaskara's life. He was "probably a Marathi astronomer". He was born at Bori, in Parbhani district of Maharashtra state in India in 7th century.

His astronomical education was given by his father. Bhaskara is considered the most important scholar of Aryabhata's astronomical school. He and Brahmagupta are the most renowned Indian mathematicians who made considerable contributions to the study of fractions.

Chandiia probably most important mathematical contribution concerns the representation of numbers in a positional system. The first positional representations were known to Indian astronomers about 500. However, the numbers were not written in figures, but in words or allegories, and were organized in verses. For instance, the number 1 was given as moon, since it exists only once; the number 2 was represented by wings, twins, or eyes, since they always occur in pairs; the number 5 was given by the (5) senses. Similar to our current decimal system, these words were aligned such that each number assigns the factor of the power of ten corresponding to its position, only in reverse order: the higher powers were right from the lower ones.

His system is truly positional, since the same words representing, can also be used to represent the values 40 or 400. Quite remarkably, he often explains a number given in this system, using the formula

circle for the zero . Contrary to his word number system, however, the figures are written in descending

Bhaskara did not invent it, but he was the first having no compunctions to use the Brahmi numerals in a scientific contribution in Sanskrit.

3.3 DIVERGENCE OF VECTOR FUNCTION

Definition : Let $\vec{f}$ is any given differentiable vector point function. Then divergence of $\vec{f}$ is written as $\nabla \cdot \vec{f}$ or $div\,\vec{f}$ is defined as

$$div\,\vec{f} = \nabla \cdot \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{f}$$

$$= \hat{i} \cdot \frac{\partial \vec{f}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{f}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{f}}{\partial z}$$

$$= \Sigma \hat{i} \cdot \frac{\partial \vec{f}}{\partial x}$$

Note: $div\,\vec{f}$ is a scalar quantity. So, divergence of vector point function is a scalar point function.

Theorem 1. If $\vec{f} = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$ is a differentiable vector point function then $\nabla \cdot \vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$.

Proof : $\nabla \cdot \vec{f}$ is defined as

$$\nabla \cdot \vec{f} = \Sigma \hat{i} \cdot \frac{\partial \vec{f}}{\partial x}$$

$$= \Sigma \hat{i} \cdot \left(\frac{\partial f_x}{\partial x}\hat{i} + \frac{\partial f_y}{\partial y}\hat{j} + \frac{\partial f_z}{\partial z}\hat{k}\right)$$

$$= \Sigma \frac{\partial f_x}{\partial x}$$

$$= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

Solenoidal Vector : A vector $\vec{f}$ is said to be solenoidal if $\nabla \cdot \vec{f} = 0$.

3.3.1 Physical Significance of Divergence

$div\,\vec{f}$ denotes source or sink of vector field $\vec{f}$. If $div\,\vec{f}$ is positive at point P in space then there exist a source of field $\vec{f}$ at that point whereas if $div\,\vec{f}$ is negative at point P in space, then there exist a sink of field $\vec{f}$ at that point.

For example: Gauss Law of electrostatics in differential form is written as $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$.

It means if $\nabla \cdot E$ is positive at some point P, then there exist source of electrostatic field i.e. positive charge and field will be diverging from that point.

If $\nabla \cdot \vec{E}$ is negative at some point P, then there exist sink of electrostatic field i.e. negative charge and field will be converging to that point. The concept of divergence is also applied in fluid dynamics.

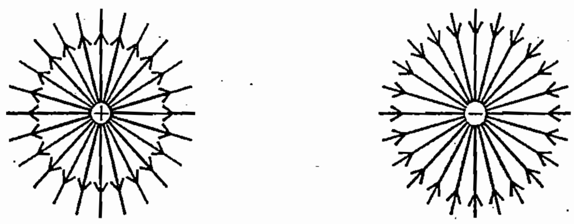


Figure 3.11

3.4 CURL OF A VECTOR POINT FUNCTION

Definition: Let $\vec{f}$ is any differentiable vector point function. Then the curl of $\vec{f}$ written as $\nabla \times \vec{f}$ is defined as

$$curl\ \vec{f} = \nabla \times \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{f}$$

$$= \hat{i} \times \frac{\partial \vec{f}}{\partial x} + \hat{j} \times \frac{\partial \vec{f}}{\partial y} + \hat{k} \times \frac{\partial \vec{f}}{\partial z}$$

$$= \sum \hat{i} \times \frac{\partial \vec{f}}{\partial x}$$

Note: $curl\ \vec{f}$ is a vector quantity. Thus the curl of a vector point for function is a vector point function.

Theorem. If $f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$ is a differentiable vector point function then

$$curl\ \vec{f} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\hat{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\hat{k}$$

Proof: Let

$$\vec{f} = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$curl\ \vec{f} = \sum \hat{i} \times \frac{\partial}{\partial x}\vec{f}$$

$$= \sum \hat{i} \times \frac{\partial}{\partial x}(f_x\hat{i} + f_y\hat{j} + f_z\hat{k})$$

$$= \sum \hat{i} \times \left(\frac{\partial f_x}{\partial x}\hat{i} + \frac{\partial f_y}{\partial x}\hat{j} + \frac{\partial f_z}{\partial x}\hat{k}\right)$$

$$= \sum \frac{\partial f_y}{\partial x}\hat{k} - \frac{\partial f_z}{\partial x}\hat{j}$$

$$= \left(\frac{\partial f_y}{\partial x}\hat{k} - \frac{\partial f_z}{\partial x}\hat{j}\right) + \left(\frac{\partial f_z}{\partial y}\hat{i} - \frac{\partial f_x}{\partial y}\hat{k}\right) + \left(\frac{\partial f_x}{\partial z}\hat{j} - \frac{\partial f_y}{\partial z}\hat{i}\right)$$

$$= \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\hat{j} + \left(\frac{\partial f_y}{\partial z} - \frac{\partial f_x}{\partial y}\right)\hat{k}$$

$curl\ \vec{f}$ can also be written as

$$curl\ \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

Irrotational Vector. A Vector $\vec{f}$ is said to be irrotational if $\nabla \times \vec{f} = 0$

Physical Significance of Curl

curl $\vec{f}$ denotes the rotational sense of a vector field $\vec{f}$.

For example: The magnetic field due to current carrying conductor is said to possess a curl.

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \text{(Ampere's law)}$$

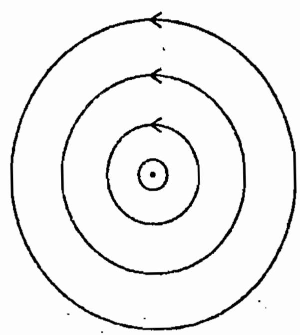


Figure 3.12

Here, magnetic field has rotational sense only. It does not converge or diverge. So, its divergence is

or diverges. Hence, the electrostatic field is irrotational.

3.5 THE LAPLACIAN OPERATOR

The Laplacian operator ∇^2 is defined as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

If f is a scalar point function then

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

$\nabla^2 f$ is also a scalar quantity.

If $\vec{f}$ is a vector point function then

$$\nabla^2 \vec{f} = \frac{\partial^2 \vec{f}}{\partial x^2}+\frac{\partial^2 \vec{f}}{\partial y^2}+\frac{\partial^2 \vec{f}}{\partial z^2}$$

$\nabla^2 \vec{f}$ is also a vector quantity.

Laplace's Equation. The equation $\nabla^2 \vec{f} = 0$ is called Laplace's equation. A function which satisfies Laplace's equation is called a harmonic function.

3.6 IMPORTANT VECTOR IDENTITIES

1. Prove that div $(\vec{A}+\vec{B}) = \text{div}\,\vec{A}+\text{div}\,\vec{B}$

or $\quad \nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A}+\nabla\cdot\vec{B}$

Proof: We have

$$\text{div}\,(\vec{A}+\vec{B}) = \nabla\cdot(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(\vec{A}+\vec{B})$$

$$= \hat{i}.\frac{\partial}{\partial x}(\vec{A}+\vec{B})+\hat{j}\cdot\frac{\partial}{\partial y}(\vec{A}+\vec{B})+\hat{k}\cdot\frac{\partial}{\partial z}(\vec{A}+\vec{B})$$

$$= \hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\right)+\hat{j}\cdot\left(\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{B}}{\partial y}\right)+\hat{k}\cdot\left(\frac{\partial\vec{A}}{\partial z}+\frac{\partial\vec{B}}{\partial z}\right)$$

$$= \left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{A}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{A}}{\partial z}\right)+\left(\hat{i}\cdot\frac{\partial\vec{B}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{B}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{B}}{\partial z}\right)$$

$$= \nabla\cdot\vec{A}+\nabla\cdot\vec{B} = \text{div}\,\vec{A}+\text{div}\,\vec{B}$$

2. Prove that curl $(\vec{A}+\vec{B}) = \text{curl}\,\vec{A}+\text{curl}\,\vec{B}$

or $\quad \nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A}+\nabla\times\vec{B}.$

Proof: We have curl $(\vec{A}+\vec{B}) = \nabla\times(\vec{A}+\vec{B})$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(\vec{A}+\vec{B})$$

$$= \sum\hat{i}\times\frac{\partial}{\partial x}(\vec{A}+\vec{B}) = \sum\hat{i}\times\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum\hat{i}\times\frac{\partial\vec{A}}{\partial x}+\sum\hat{i}\times\frac{\partial\vec{B}}{\partial x} = \text{curl}\,\vec{A}+\text{curl}\,\vec{B}.$$

3. If $\vec{A}$ is a differentiable vector function and is a differentiable scalar function, then

$$\operatorname{div}(\phi\vec{A}) = (\operatorname{grad}\phi)\cdot\vec{A} + \phi \operatorname{div}\vec{A}$$

or $$\nabla\cdot(\phi\vec{A}) = (\nabla\phi)\cdot\vec{A} + \phi(\nabla\cdot\vec{A}).$$

Proof: We have

$$\operatorname{div}(\phi\vec{A}) = \nabla\cdot(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\phi\vec{A})$$

$$= \hat{i}\cdot\frac{\partial}{\partial x}(\phi\vec{A}) + \hat{j}\cdot\frac{\partial}{\partial y}(\phi\vec{A}) + \hat{k}\cdot\frac{\partial}{\partial z}(\phi\vec{A})$$

$$= \sum\left\{\hat{i}\cdot\frac{\partial}{\partial x}(\phi\vec{A})\right\}$$

$$= \sum\left\{\hat{i}\cdot\left(\frac{\partial\phi}{\partial x}\vec{A} + \phi\frac{\partial\vec{A}}{\partial x}\right)\right\}$$

$$= \sum\left\{\hat{i}\cdot\left(\frac{\partial\phi}{\partial x}\vec{A}\right)\right\} + \sum\left\{\hat{i}\cdot\left(\phi\frac{\partial\vec{A}}{\partial x}\right)\right\}$$

$$= \sum\left\{\left(\frac{\partial\phi}{\partial x}\right)(\hat{i}\cdot\vec{A})\right\} + \sum\left\{\phi\left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)\right\}$$

[Note $\vec{a}.(m\vec{b}) = (m\vec{a}).\vec{b} = m(\vec{a}\cdot\vec{b})$]

$$= \left\{\sum\hat{i}\frac{\partial\phi}{\partial x}\right\}\cdot\vec{A} + \phi\sum\left\{\left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)\right\} = (\nabla\phi)\cdot\vec{A} + \phi(\nabla\cdot\vec{A})$$

4. Prove that curl $(\phi\vec{A}) = (\operatorname{grad}\phi)\times\vec{A} + \phi\operatorname{curl}\vec{A}$

or $$\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A} - \phi(\nabla\times\vec{A}).$$

Proof: We have

$$\operatorname{curl}(\phi\vec{A}) = \nabla\times(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(\phi\vec{A})$$

$$= \Sigma\left\{\hat{i}\times\frac{\partial}{\partial x}(\phi\vec{A})\right\} = \Sigma\left\{\hat{i}\times\left(\frac{\partial\phi}{\partial x}\vec{A} + \phi\frac{\partial\vec{A}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{\hat{i}\times\left(\frac{\partial\phi}{\partial x}\vec{A}\right)\right\} + \Sigma\left\{\hat{i}\times\left(\phi\frac{\partial\vec{A}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{\left(\hat{i}\frac{\partial\phi}{\partial x}\right)\times\vec{A}\right\} + \Sigma\left\{\phi\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right)\right\}$$

[Note that $\vec{a}\times(m\vec{b}) = (m\vec{a})\times\vec{b} = m(\vec{a}\times\vec{b})$]

$$= \left\{\Sigma\left(\hat{i}\frac{\partial\phi}{\partial x}\right)\right\}\times\vec{A} + \phi\Sigma\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A})$$

5. Prove that div $(\vec{A}\times\vec{B}) = \vec{B}\cdot\operatorname{curl}\vec{A} - \vec{A}\cdot\operatorname{curl}\vec{B}$

or $$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

Proof: We have

$$\operatorname{div}(\vec{A}\times\vec{B}) = \Sigma\left\{\hat{i}\cdot\frac{\partial}{\partial x}(\vec{A}\times\vec{B})\right\} = \Sigma\left\{\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B} + \vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\hat{i}\cdot\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B}\right)\right\}+\Sigma\left\{\hat{i}\cdot\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\left(\hat{i}\times\frac{\partial \vec{A}}{\partial x}\right)\cdot\vec{B}\right\}-\Sigma\left\{\hat{i}\cdot\left(\frac{\partial \vec{B}}{\partial x}\times\vec{A}\right)\right\}$$

[Note $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$ and $\vec{a}\cdot(\vec{b}\times\vec{c})=-\vec{a}\cdot(\vec{c}\times\vec{b})$]

$$=\left\{\Sigma\left(\hat{i}\times\frac{\partial \vec{A}}{\partial x}\right)\right\}\cdot\vec{B}-\Sigma\left\{\left(\hat{i}\times\frac{\partial \vec{B}}{\partial x}\right)\cdot\vec{A}\right\}$$

$$=(\text{curl }\vec{A})\cdot\vec{B}-\left\{\Sigma\left(\hat{i}\times\frac{\partial \vec{B}}{\partial x}\right)\right\}\cdot\vec{A}$$

$$=(\text{curl }\vec{A})\cdot\vec{B}-(\text{cul }\vec{B})\cdot\vec{A}$$

$$=\vec{B}\cdot\text{curl }\vec{A}-\vec{A}\cdot\text{curl }\vec{B}.$$

6. Prove that

$$\text{curl }(\vec{A}\times\vec{B})=(\vec{B}\cdot\nabla)\ \vec{A}-\vec{B}\text{ div }\vec{A}-(\vec{A}\cdot\nabla)\ \vec{B}+\vec{A}\text{ div }\vec{B}.$$

Proof: We have

$$\text{curl }(\vec{A}\times\vec{B})=\nabla\times(\vec{A}\times\vec{B})$$

$$=\Sigma\left\{\hat{i}\times\frac{\partial}{\partial x}(\vec{A}\times\vec{B})\right\}=\Sigma\left\{\hat{i}\times\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x}+\frac{\partial \vec{A}}{\partial x}+\vec{B}\right)\right\}$$

$$=\Sigma\left\{\hat{i}\times\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x}\right)\right\}+\Sigma\left\{\hat{i}\times\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B}\right)\right\}$$

$$=\Sigma\left\{\left(\hat{i}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{A}-(\hat{i}\cdot\vec{A})\frac{\partial \vec{B}}{\partial x}\right\}+\Sigma\left\{(\hat{i}\cdot\vec{B})\frac{\partial \vec{A}}{\partial x}-\left(\hat{i}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right\}$$

$$=\Sigma\left\{\left(\hat{i}\cdot\frac{\partial \vec{B}}{\partial x}\vec{A}\right)\right\}-\Sigma\left\{(\vec{A}\cdot\hat{i})\frac{\partial \vec{B}}{\partial x}\right\}+\Sigma\left\{(\vec{B}\cdot\hat{i})\frac{\partial \vec{A}}{\partial x}\right\}-\Sigma\left\{\left(\hat{i}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right\}$$

$$=\Sigma\left\{\left(\hat{i}\cdot\frac{\partial \vec{B}}{\partial x}\right)\right\}\vec{A}-\left\{\vec{A}\cdot\Sigma\hat{i}\frac{\partial}{\partial x}\right\}\vec{B}+\left\{\vec{B}\cdot\Sigma\hat{i}\frac{\partial}{\partial x}\right\}\vec{A}-\left\{\Sigma\left(\hat{i}\cdot\frac{\partial \vec{A}}{\partial x}\right)\right\}\vec{B}$$

$$=(\text{div }\vec{B})\ \vec{A}-(\vec{A}\cdot\nabla)\vec{B}+(\vec{B}\cdot\nabla)\vec{A}-(\text{div }\vec{A})\vec{B}.$$

7. Prove that

$$\text{grad}(\vec{A}\cdot\vec{B})=(\vec{B}\cdot\nabla)\vec{A}+(\vec{A}\cdot\nabla)\vec{B}+\vec{B}\times\text{curl }\vec{A}+\vec{A}\times\text{curl }\vec{B}.$$

Proof: We have

$$\text{grad}(\vec{A}\cdot\vec{B})=\nabla(\vec{A}\cdot\vec{B})$$

$$=\Sigma\hat{i}\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B})$$

$$=\Sigma\hat{i}\left(\vec{A}\cdot\frac{\partial \vec{B}}{\partial x}+\frac{\partial \vec{A}}{\partial x}\cdot\vec{B}\right)$$

$$=\Sigma\left\{\left(\vec{A}\cdot\frac{\partial \vec{B}}{\partial x}\right)\hat{i}\right\}+\Sigma\left\{\left(\vec{B}\cdot\frac{\partial \vec{A}}{\partial x}\right)\hat{i}\right\} \quad .....(1)$$

Now we know that $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$\therefore \quad (\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - \vec{a}\times(b\times\vec{c})$

$$\therefore \quad \left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i} = (\vec{A}\cdot\hat{i})\frac{\partial\vec{B}}{\partial x} - \vec{A}\times\left(\frac{\partial\vec{B}}{\partial x}\times\hat{i}\right)$$

$$= (\vec{A}\cdot\hat{i})\frac{\partial\vec{B}}{\partial x} + \vec{A}\times\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)$$

Then
$$\Sigma\left\{\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i}\right\} = \Sigma\left\{(\vec{A}\cdot\hat{i})\frac{\partial\vec{B}}{\partial x}\right\} + \Sigma\left\{\vec{A}\times\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)\right\}$$

$$= \left\{\vec{A}\cdot\Sigma\hat{i}\frac{\partial}{\partial x}\right\}\vec{B} + \vec{A}\times\Sigma\left\{\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right\}$$

$$= (\vec{A}\cdot\nabla)\vec{B} + \vec{A}\times(\nabla\times\vec{B}) \qquad .....(2)$$

Similarly, $$\Sigma\left\{\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i}\right\} = (\vec{B}\cdot\nabla)\cdot\vec{A} + \vec{B}\times(\nabla\times\vec{A}). \qquad .....(3)$$

Putting the values from (2) and (3) in (1), we get

$$\text{grad }(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\,\vec{B} + \vec{A}\times(\nabla\times\vec{B}) + (\vec{B}\cdot\nabla)\,\vec{A} + \vec{B}\times(\nabla\times\vec{A}).$$

Note: If we put $\vec{A}$ in place of $\vec{B}$. then

$$\text{grad }(\vec{A}\cdot\vec{A}) = 2(\vec{A}\cdot\nabla)\vec{A} + 2\vec{A}\times(\nabla\times\vec{A}).$$

or
$$\frac{1}{2}\text{grad } A^2 = ((\vec{A}\cdot\nabla)\vec{A} + \vec{A}\times\text{curl } A.)$$

8. Prove that div grad $\phi = \nabla^2\phi$

i.e. $\quad \nabla\cdot(\nabla\phi) = \nabla^2\phi$

Proof: We have

$$\nabla\cdot(\nabla\phi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial\phi}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

9. Prove that curl of the gradient of ϕ is zero

$$\nabla\times(\nabla\phi) = 0, \text{ i.e. curl grad } \phi = 0.$$

Proof: We have

$$\text{grad }\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\therefore \quad \text{curl grad }\phi = \nabla\times\text{grad }\phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right)\hat{i} + \left(\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z}\right)\hat{j} + \left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0,$$

provided we suppose that has continuous second partial derivatives so that the order of the differentiation is immaterial.

10. Prove that div curl $\vec{A} = 0$, *i.e.*, $\nabla\cdot(\nabla\times\vec{A}) = 0$

Proof: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$.

Then curl $\vec{A} = \nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

Now div curl $\vec{A} = \nabla\cdot(\nabla\times\vec{A})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x\,\partial y} - \frac{\partial^2 A_2}{\partial x\,\partial z} + \frac{\partial^2 A_1}{\partial y\,\partial z} - \frac{\partial^2 A_3}{\partial y\,\partial x} + \frac{\partial^2 A_2}{\partial z\,\partial x} - \frac{\partial^2 A_1}{\partial z\,\partial y}$$

$= 0$, assuming that $\vec{A}$ has continuous second partial derivatives.

11. Prove that

$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

Proof: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$.

Then $\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

$$\therefore \quad \nabla\times(\nabla\times\vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right\}\hat{i}\right]$$

$$= \Sigma\left[\left\{\left(\frac{\partial^2 A_2}{\partial x\,\partial y} + \frac{\partial^2 A_3}{\partial z\,\partial x}\right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}\hat{i}\right]$$

$$= \sum\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\hat{i}\right]$$

$$= \sum\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\hat{i}\right]$$

$$= \sum\left[\left\{\frac{\partial}{\partial x}(\nabla\cdot\vec{A})-(\nabla^2 A_1)\right\}\hat{i}\right]$$

$$= \sum\left[\left\{\frac{\partial}{\partial x}(\nabla\cdot\vec{A})\right\}\hat{i}\right]-\nabla^2\sum A_1\hat{i}=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$$

SOLVED EXAMPLES (OBJECTIVE)

1. $div\,\vec{r}$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (d)

$$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$\nabla\cdot\vec{r} = \sum\hat{i}\cdot\frac{\partial}{\partial x}\vec{r}$$

$$= \sum\hat{i}\cdot\hat{i}=\sum 1=3$$

2. curl $\vec{r}$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (a)

$$\text{curl}\,\vec{r} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}$$

$$=0$$

3. The value of constant a for which the vector $\vec{f}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ is solenoidal is

(a) 1 (b) 2 (c) –2 (d) –1

Ans. (c)

Vector $\vec{f}$ is solenoidal if $div\,\vec{f}=0$.

$$\text{div}\,\vec{f} = \frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+az)=0$$

$$\Rightarrow \quad 1+1+a=0$$

$$a=-2$$

4. If $\vec{a}$ is a constant vector, then $\nabla\cdot(\vec{r}\times\vec{a})$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (a)

$$div(\vec{r}\times\vec{a}) = \sum\hat{i}\cdot\frac{\partial}{\partial x}(\vec{r}\times\vec{a})$$

$$= \sum\hat{i}\cdot\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right)$$

$$= \sum\hat{i}\cdot(\hat{i}\times a)=0$$

5. If $\vec{a}$ is a constant vector, curl $(\vec{r} \times \vec{a})$ is equal to

(a) $\vec{a}$ (b) $2\vec{a}$ (c) $-2\vec{a}$ (d) $-\vec{a}$

Ans. (c)

$$\text{curl } (\vec{r} \times \vec{a}) = \sum \hat{i} \times \frac{\partial}{\partial x}(\vec{r} \times \vec{a})$$

$$= \sum \hat{i} \times \left(\frac{\partial \vec{r}}{\partial x} \times \vec{a}\right)$$

$$= \sum \hat{i} \times (\hat{i} \times a)$$

$$= \sum [(\hat{i} \cdot \vec{a})\hat{i} - (\hat{i} \cdot \hat{i})\vec{a}]$$

$$= \sum (\hat{i} \cdot \vec{a})\hat{i} - \sum \vec{a}$$

$$= \vec{a} - 3\vec{a} = -2\vec{a}$$

6. If $\vec{f} = e^{xyz}(\hat{i} + 2\hat{j} + 3\hat{k})$ the curl $\vec{f}$ at (1, 1, 1) equal to

(a) $(\hat{i} + \hat{j} + \hat{k})$ (b) $e(\hat{i} + \hat{j})$ (c) $e(\hat{i} + \hat{k})$ (d) $e(\hat{j} + \hat{k})$

Ans. (c)

$$\nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} & 2e^{xyz} & 3e^{xyz} \end{vmatrix}$$

$$= e^{xyz}(3xz - 2xy)\hat{i} + e^{xyz}(xy - yz)\hat{j} + e^{xyz}(2yz - xz)\hat{k}$$

At (1, 1, 1),

$$\nabla \times \vec{f} = e(\hat{i} + \hat{k})$$

7. If $\vec{f} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$, then value of $div\,\vec{f}$ at (1, 1, 1) is equal to

(a) 0 (b) –1 (c) –2 (d) –3

Ans. (d)

$$\vec{f} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k},$$

$$\text{div } \vec{f} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) - \frac{\partial}{\partial z}(3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

At (1, 1, 1), $\text{div } \vec{f} = -3$

8. If $\vec{f} = (x^2 - y^2)\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$, the curl $\vec{f}$ at (1, 1, 1) is equal to

(a) $\hat{i} + \hat{j} + \hat{k}$ (b) $\hat{i} + 2\hat{j} + 3\hat{k}$ (c) $\hat{i} + \hat{j} + 4\hat{k}$ (d) $\hat{i} - \hat{j} - \hat{k}$

Ans. (c)

$$\vec{f} = (x + y + 1)\hat{i} + \hat{j} + (-x - y)\hat{k}$$

$$\text{curl } \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + 1 & 1 & -x - y \end{vmatrix}$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

$$\vec{f}\cdot curl\,\vec{f} = ((x+y+1)\hat{i}+\hat{j}+(-x-y)\hat{k})\cdot(-\hat{i}+\hat{j}-\hat{k})$$
$$= -x-y-1+1+x+y=0$$

SOLVED EXAMPLES (SUBJECTIVE)

1. Prove that $div\,(r^n\vec{r}) = (n+3)r^n$

Solution. $div = (r^n\vec{r})$

$$= \sum \hat{i}.\frac{\partial}{\partial x}(r^n\vec{r})$$
$$= \sum \hat{i}.\left[nr^{n-1}\frac{\partial r}{\partial x}\vec{r}+r^n\frac{\partial \vec{r}}{\partial x}\right]$$
$$= \sum\left[nr^{n-1}\frac{x}{r}(\hat{i}.\vec{r})+r^n\hat{i}.\hat{i}\right]\left(\frac{\partial r}{\partial x}=\frac{x}{r},\frac{\partial \vec{r}}{\partial x}=\hat{i}\right)$$
$$= nr^{n-2}\Sigma x^2 + r^n\,\Sigma 1$$
$$= nr^n + 3r^n$$
$$= (n+3)r^n$$

2. Prove that $\nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\vec{r}$

Solution. $\nabla^2(r^n\vec{r}) = \nabla(\nabla\cdot(r^n\vec{r}))$

$\nabla\cdot(r^n\vec{r}) = (n+3)r^n$ (from previous example)

So, $\nabla^2(r^n\vec{r}) = \nabla\,(n+3)r^n$

$$= (n+3)\sum \hat{i}\,\frac{\partial(r^n)}{\partial x}$$
$$= (n+3)\sum nr^{n-1}\hat{i}\,\frac{\partial r}{\partial x}$$
$$= n(n+3)r^{n-2}\,\Sigma x\hat{i}$$
$$= n(n+3)r^{n-2}\,\vec{r}$$

3. Prove that $div\left(\frac{\vec{r}}{r^3}\right)=0$

Solution. $div\left(\frac{\vec{r}}{r^3}\right) = \sum \hat{i}\cdot\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^3}\right)$

$$= \sum \hat{i}\cdot\left[\frac{1}{r^3}\frac{\partial \vec{r}}{\partial x}+\vec{r}\,\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)\right]$$
$$= \sum \hat{i}\cdot\left[\frac{1}{r^3}\hat{i}+\vec{r}\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}\right)\right]$$
$$= \frac{1}{r^3}\Sigma\hat{i}\cdot\hat{i}-\frac{3}{r^5}\Sigma(\hat{i}\cdot\vec{r})x$$
$$= \frac{1}{r^3}\Sigma 1-\frac{3}{r^5}\Sigma x^2$$
$$= \frac{3}{r^3}-\frac{3}{r^5}\cdot r^2$$
$$= 0$$

4. Prove that $div\ \hat{e}_r = \frac{2}{r}$

Solution. $\nabla\cdot\hat{e}_r = \nabla\cdot\left(\frac{\vec{r}}{r}\right)$

$$= \sum \hat{i} \cdot \frac{\partial}{\partial x}\left(\frac{\bar{r}}{r}\right)$$

$$= \sum \hat{i} \cdot \left(\frac{1}{r}\frac{\partial \bar{r}}{\partial x} + \bar{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right)$$

$$= \sum \hat{i} \cdot \left(\frac{1}{r}\hat{i} + \bar{r}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right)$$

$$= \sum \left(\frac{1}{r}\hat{i}\cdot\hat{i} - \frac{1}{r^2}\cdot\frac{x}{r}(\hat{i}\cdot\bar{r})\right)$$

$$= \frac{1}{r}\cdot\Sigma 1 - \frac{1}{r^3}\Sigma x^2$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

5. Prove that vector $f(r)\,\bar{r}$ is irrotational

Solution. A vector function is said to be irrotational if its curl is zero.

$$\nabla \times (f(r)\bar{r}) = \sum \hat{i} \times \frac{\partial}{\partial x}(f(r)\bar{r})$$

$$= \sum \hat{i} \times \left(f'(r)\frac{\partial r}{\partial x}\bar{r} + f(r)\frac{\partial \bar{r}}{\partial x}\right)$$

$$= \sum \hat{i} \times \left(f'(r)\frac{x}{r}\bar{r} + f(r)\hat{i}\right)$$

$$= \frac{f'(r)}{r}\Sigma x\hat{i} \times \bar{r} + \Sigma f(r)\Sigma \hat{i} \times \hat{i}$$

$$= \frac{f'(r)}{r}\bar{r} \times \bar{r} + f(r)\Sigma \hat{i} \times \hat{i}$$

$$= 0$$

Since, curl of $f(r)\,\bar{r}$ is zero, hence, $f(r)\,\bar{r}$ is irrotational.

6. Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$

Solution. $$\nabla^2\left(\frac{1}{r}\right) = \nabla \cdot \left(\nabla \frac{1}{r}\right)$$

$$\nabla\left(\frac{1}{r}\right) = \sum \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \sum \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)$$

$$= \sum \hat{i}\left(-\frac{1}{r^2}\frac{x}{r}\right)$$

$$= -\frac{1}{r^3}\Sigma x\hat{i} = -\frac{\bar{r}}{r^3}$$

$$\nabla^2\left(\frac{1}{r}\right) = \nabla \cdot \left(\nabla \frac{1}{r}\right)$$

$$= \nabla \cdot \left(-\frac{\bar{r}}{r^3}\right)$$

$$= \sum \hat{i} \cdot \frac{\partial}{\partial x}\left(-\frac{\vec{r}}{r^3}\right)$$

$$= -\sum \hat{i} \cdot \left(\frac{1}{r^3}\frac{\partial \vec{r}}{\partial x} + \vec{r}\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)\right)$$

$$= -\sum \hat{i} \cdot \left(\frac{1}{r^3}\hat{i} + \vec{r}\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}\right)\right)$$

$$= -\sum\left(\frac{1}{r^3}(\hat{i}\cdot\hat{i}) - \frac{3x}{r^5}\cdot(\hat{i}\cdot\vec{r})\right)$$

$$= -\frac{1}{r^3}\Sigma 1 + \frac{3}{r^5}\Sigma x^2$$

$$= -\frac{3}{r^3} + \frac{3}{r^5}\cdot r^2 = 0$$

7. Prove that div grad $r^n = n(n+1)r^{n-2}$

Solution.

$$\text{grad } r^n = \sum \hat{i}\frac{\partial}{\partial x}r^n$$

$$= \sum \hat{i} n r^{n-1}\frac{\partial r}{\partial x}$$

$$= \sum \hat{i} n r^{n-1}\frac{x}{r}$$

$$= n r^{n-2}\Sigma x\hat{i}$$

$$= n r^{n-2}\vec{r}$$

$$div \text{ grad } r^n = \sum div(n r^{n-2}\vec{r})$$

$$= \sum \hat{i}\cdot\frac{\partial}{\partial x}(n r^{n-2}\vec{r})$$

$$= n\sum \hat{i}.\left(r^{n-2}\frac{\partial \vec{r}}{\partial x} + \vec{r}\frac{\partial}{\partial x}(r^{n-2})\right)$$

$$= n\sum \hat{i}.\left(r^{n-2}\hat{i} + \vec{r}(n-2)r^{n-3}\frac{\partial r}{\partial x}\right)$$

$$= n r^{n-2}\Sigma\hat{i}\cdot\hat{i} + n\Sigma\hat{i}.\left((n-2)r^{n-3}\frac{x}{r}\vec{r}\right)$$

$$= 3n r^{n-2} + n(n-2)r^{n-4}\Sigma x(\hat{i}.\vec{r})$$

$$= 3n r^{n-2} + n(n-2)r^{n-4}\Sigma x^2$$

$$= 3n r^{n-2} + n(n-2)r^{n-2}$$

$$= (n^2+n)r^{n-2}$$

$$= n(n+1)r^{n-2}$$

8. Prove that $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\cdot\nabla\psi + \psi\nabla^2\phi$

Solution.

$$\nabla^2(\phi\psi) = \nabla\cdot(\nabla(\phi\psi))$$

$$= \nabla\cdot(\psi\nabla\phi + \phi\nabla\psi)$$

$$= \nabla\cdot(\psi\nabla\phi) + \nabla\cdot(\phi\nabla\psi)$$

$$= \psi\nabla^2\phi + 2\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi$$

9. If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A}\times\vec{B}$ is solenaidal

Solution. $\vec{A}\times\vec{B}$ are irrotational

So, $\nabla\times\vec{A}=0$. & $\nabla\times\vec{B}=0$

Now,
$$\nabla\cdot(\vec{A}\times\vec{B}) = \sum\hat{i}\cdot\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$
$$= \sum\left[\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)+\hat{i}\cdot\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right]$$
$$= \sum\left[\vec{B}\cdot\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right)-\hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)\right]$$
$$= \vec{B}\cdot\sum\hat{i}\times\frac{\partial\vec{A}}{\partial x}-\sum\hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)=\vec{B}\cdot\text{curl}\,\vec{A}-\vec{A}\cdot\sum\hat{i}\times\frac{\partial\vec{B}}{\partial x}$$
$$= \vec{B}\cdot\text{curl}\,\vec{A}-\vec{A}\,\text{curl}\,\vec{B}=0$$

Since, $\nabla\cdot(\vec{A}\times\vec{B})=0$

Hence, $\vec{A}\times\vec{B}$ is solenoidal.

10. If f and g are two scalar point function, prove that div $(f\nabla g)=f\nabla^2 g+\nabla f\cdot\nabla g$.

Solution. We can use a vector identity
$$\nabla\cdot(\phi\vec{f})=\nabla\phi\cdot\vec{f}+\phi\nabla\cdot\vec{f}$$
Where ϕ is a scalar function & $\vec{f}$ is a vector function

So,
$$\nabla\cdot(f\nabla g)=\nabla f\cdot\nabla g+f\nabla\cdot(\nabla g)$$
$$= \nabla f\cdot\nabla g+f\nabla^2 g$$

Aliter:
$$f\nabla g = f\left(\sum\hat{i}\frac{\partial}{\partial x}g\right)$$
$$= f\frac{\partial g}{\partial x}\hat{i}+f\frac{\partial g}{\partial y}\hat{j}+f\frac{\partial g}{\partial z}\hat{k}$$
$$\nabla\cdot(f\nabla g)=\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$
$$= \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial x^2}+\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial y^2}+\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial z}+f\frac{\partial^2 g}{\partial z^2}$$
$$= f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial x}\right)$$
$$= f\nabla^2 g+\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial g}{\partial x}\hat{i}+\frac{\partial g}{\partial y}\hat{j}+\frac{\partial g}{\partial z}\hat{k}\right)$$
$$= f\nabla^2 g+\nabla f\cdot\nabla g$$

11. Prove that div $(\vec{A}\times\vec{r})$ = $\vec{r}$.curl $\vec{A}$ when $\vec{A}$ is a constant vector

Solution.

Using identity $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\text{curl}\,\vec{A}-\vec{A}\cdot\text{curl}\,\vec{B}$
$$\nabla\cdot(\vec{A}\times\vec{r})=\vec{r}\cdot\text{curl}\,\vec{A}-\vec{A}\cdot\text{curl}\,\vec{r}$$
$$= \vec{r}\cdot\text{curl}\,\vec{A}\qquad(\text{as curl}\,\vec{r}=0)$$

12. If $\vec{a}$ is a constant vector, prove that

$$\text{div } \{r^n(\vec{a}\times\vec{r})\} = 0.$$

Solution.

$$\begin{aligned}
\nabla\cdot(r^n(\vec{a}\times\vec{r}) &= \sum \hat{i}\cdot\frac{\partial}{\partial x}\left(r^n(\vec{a}\times\vec{r})\right)\\
&= \sum \hat{i}\cdot\left(nr^{n-1}\frac{\partial r}{\partial x}(\vec{a}\times\vec{r}) + r^n\left(\vec{a}\times\frac{\partial \vec{r}}{\partial x}\right)\right)\\
&= nr^{n-1}(\Sigma xi)\cdot(\vec{a}\times\vec{r}) + r^n\,\Sigma\hat{i}.(\vec{a}\times\hat{i})\\
&= nr^{n-2}\,\vec{r}\cdot(\vec{a}\times\vec{r}) + r^n\,\Sigma\hat{i}.(\vec{a}\times\hat{i}) = 0
\end{aligned}$$

13. Prove that

$$\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$$

Solution.

$$\begin{aligned}
\nabla\cdot(\phi\nabla\psi) &= \phi\nabla\cdot(\nabla\psi) + \nabla\phi\cdot\nabla\psi\\
&= \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\\
\nabla\cdot(\psi\nabla\phi) &= \psi\nabla\cdot(\nabla\phi) + \nabla\psi\cdot\nabla\phi\\
&= \psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi
\end{aligned}$$

So,

$$\begin{aligned}
\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) &= (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi) - (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)\\
&= \phi\nabla^2\psi - \psi\nabla^2\phi
\end{aligned}$$

14. If $\vec{a}$ and $\vec{b}$ are constant vectors, prove that

(i) div $[(\vec{r}\times\vec{a})\times\vec{b}] = -2\,\vec{b}\cdot\vec{a}$

(ii) curl $[(\vec{r}\times\vec{a})\times\vec{b}] = \vec{b}\times\vec{a}$

Solution.

(i)

$$\begin{aligned}
\text{div}[(\vec{r}\times\vec{a})\times\vec{b}] &= \nabla\cdot\left[(\vec{r}\times\vec{a})\times\vec{b}\right]\\
&= \sum \hat{i}\cdot\frac{\partial}{\partial x}\left[(\vec{r}\times\vec{a})\times\vec{b}\right]\\
&= \sum \hat{i}\cdot\left[\left(\frac{\partial \vec{r}}{\partial x}\times\vec{a}\right)\times\vec{b}\right]\\
&= \sum \hat{i}\cdot\left[(\hat{i}\times\vec{a})\times\vec{b}\right]\\
&= \sum \hat{i}\cdot\left[(\hat{i}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{b})\hat{i}\right]\\
&= \sum\left[(\hat{i}\cdot\vec{b})(\hat{i}\cdot\vec{a}) - (\vec{a}\cdot\vec{b})(\hat{i}\cdot\hat{i})\right]\\
&= \sum a_x b_x - (\vec{a}\cdot\vec{b})\Sigma 1\\
&= \vec{a}\cdot\vec{b} - 3(\vec{a}\cdot\vec{b})\\
&= -2\,\vec{a}\cdot\vec{b} = -2\,\vec{b}\cdot\vec{a}
\end{aligned}$$

(ii)

$$\begin{aligned}
\text{curl }[(\vec{r}\times\vec{a})\times\vec{b}] &= \sum \hat{i}\times\frac{\partial}{\partial x}\left[(\vec{r}\times\vec{a})\times\vec{b}\right]\\
&= \sum \hat{i}\times\left[\left(\frac{\partial \vec{r}}{\partial x}\times\vec{a}\right)\times\vec{b}\right]\\
&= \sum \hat{i}\times\left[(\hat{i}\times\vec{a})\times\vec{b}\right]\\
&= \sum \hat{i}\times\left[(\hat{i}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{b})\hat{i}\right]
\end{aligned}$$

$$= \sum(\hat{i}\cdot\vec{b})(i\times\vec{a})-(\vec{a}\cdot\vec{b})(\hat{i}\times\hat{i})$$
$$= \sum b_x(a_y\hat{k}-a_z\hat{j})$$
$$= (b_xa_y\hat{k}-b_xa_z\hat{j})+(b_ya_z\hat{i}-b_ya_x\hat{k})+(b_za_x\hat{j}-b_za_y\hat{i})$$
$$= (b_ya_z-b_za_y)\hat{i}+(b_za_x-b_xa_z)\hat{j}+(b_xa_y-b_ya_x)\hat{k}$$
$$= \vec{b}\times\vec{a}$$

15. If $\vec{r}$ denotes the position vector of a point and if $\hat{r}$ be the unit vector in the direction of $\vec{r}$, $r=|\vec{r}|$, determine of grad (r^{-1}) in terms of $\hat{r}$ and r.

Solution.
$$\nabla\left(\frac{1}{r}\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$
$$=\sum\hat{i}\left(-\frac{1}{r^2}\right).\frac{\partial r}{\partial x}$$
$$=-\frac{1}{r^3}\sum x\hat{i}$$
$$=-\frac{\vec{r}}{r^3}$$
$$\nabla\left(\frac{1}{r}\right)=\frac{\hat{r}}{r^2}$$

Where, $\hat{r}$ is a unit vector in direction of $\vec{r}$

16. For any constant vector $\vec{a}$, show that the vector represented by curl $(\vec{a}\times\vec{r})$ is always parallel to the vector $\vec{a}$, $\vec{r}$ being the position vector of a point (x, y, z), measured from the origin.

Solution.
$$\text{Curl }(\vec{a}\times\vec{r})=\sum\hat{i}\times\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)$$
$$=\sum\hat{i}\times\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)$$
$$=\sum\hat{i}\times(\vec{a}\times\hat{i})=\sum\hat{i}\times(\vec{a}\times\hat{i})$$
$$=\sum(\hat{i}\cdot\hat{i})\vec{a}-(\hat{i}\cdot\vec{a})\hat{i}$$
$$=\vec{a}\sum 1-\sum(\hat{i}\cdot\vec{a})\hat{i}$$
$$=3\vec{a}-\vec{a}=2\vec{a}$$

So, Curl $(\vec{a}\times\vec{r})$ is always parallel to vector $\vec{a}$

17. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, find the value(s) of n in order that $r^n\vec{r}$ may be (i) solenoidal, (ii) irrotational.

Solution.

(i) For vector to be solenoidal, its divergence should be zero.

$$\nabla.(r^n\vec{r})=\sum\hat{i}.\frac{\partial}{\partial x}(r^n\vec{r})$$
$$=\sum\left[(\hat{i}.\vec{r})nr^{n-1}\frac{\partial r}{\partial x}+r^n\hat{i}.\frac{\partial\vec{r}}{\partial x}\right]$$
$$=\sum\left[nr^{n-1}\frac{x^2}{r}+r^n(\hat{i}.\hat{i})\right]$$
$$=nr^{n-2}\sum x^2+r^n\sum 1$$
$$=nr^n+3r^n=(n+3)\,r^n$$

So, $r^n\vec{r}$ will solenoidal if $n=-3$

(ii) For vector to be irrotational, its curl must be zero.

$$\text{Curl } r^n\vec{r} = \sum \hat{i}\times\frac{\partial}{\partial x}(r^n\vec{r})$$
$$= \sum\left[(\hat{i}\times\vec{r})nr^{n-1}\frac{\partial r}{\partial x}+r^n(\hat{i}\times\frac{\partial\vec{r}}{\partial x})\right]$$
$$= \sum\left[(\hat{i}\times\vec{r})nr^{n-2}x+r^n(\hat{i}\times\hat{i})\right]$$
$$= \sum\left[nr^{n-2}(x\hat{i}\times\vec{r})+0\right]$$
$$= nr^{n-2}\left[(\sum xi)\times\vec{r}\right]$$
$$= nr^{n-2}\left[\vec{r}\times\vec{r}\right]=0$$

So, $r^n\vec{r}$ will be irrotational for any value of n.

18. If $u\vec{f}=\nabla v$, when u, v are scalar fields and $\vec{f}$ is a vector, find the value of $\vec{f}$. curl $\vec{f}$.

Solution.

Given $u\vec{f}=\nabla V$

Taking curl on both sides

$$\nabla\times(u\vec{f})=\nabla\times(\nabla V)$$

$\Rightarrow \quad \nabla u\times\vec{f}+u\nabla\times\vec{f}=0$ (as curl of gradient $=0$)

Taking dot product with $\vec{f}$

$$\vec{f}.(\nabla u\times\vec{f})+u\vec{f}.\nabla\times\vec{f}=0$$

Since $\vec{f}.(\nabla u\times\vec{f})=0$

So, $\vec{f}.\nabla\times\vec{f}=0$

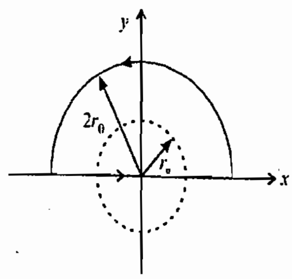
19. Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $r=|\vec{r}|$. If a scalar field φ and a vector field $\vec{u}$ satisfy $\nabla\varphi=\nabla\times\vec{u}+f(r)\vec{r}$ where f is an arbitrary differentiable function, then show that $\nabla^2\varphi=rf'(r)+3f(r)$.

Solution.

$$\nabla\phi=\nabla\times\vec{u}+f(r)\vec{r}$$
$$\nabla^2\phi=\nabla\cdot\nabla\phi$$
$$=\nabla\cdot(\nabla\times\vec{u})+\nabla\cdot(f(r)\vec{r})$$
$$=0+\sum\hat{i}\cdot\frac{\partial}{\partial x}(f(r)\vec{r})$$
$$=\sum(\hat{i}\cdot\vec{r})f'(r)\frac{x}{r}+\sum\hat{i}f(r)\cdot\frac{\partial\vec{r}}{\partial x}$$
$$=\frac{f'(r)}{r}\sum x^2+f(r)\sum\hat{i}\cdot\hat{i}$$
$$=rf'(r)+3f(r)$$

20. A vector field is given by

$$\vec{F}(r)=\begin{cases}\alpha(x\hat{j}-y\hat{i}) \\ \qquad for\ (x^2+y^2)\le r_0^2\ (\text{region}-I) \\ \alpha r_0^2\dfrac{(x\hat{j}-y\hat{i})}{(x^2+y^2)} \\ \qquad for\ (x^2+y^2)>r_0^2\ (\text{region}-II)\end{cases}$$



Here a and r_0 are two constants.
Find the curl of this field in both the region

Solution.

$$\vec{F}(r) = \alpha(-y\hat{i} + x\hat{j}) \qquad \text{for } r \le r_o \text{ (Region I)}$$

$$\alpha r_o^2 \frac{(-y\hat{i} + x\hat{j})}{x^2 + y^2} \qquad \text{for } r > r_o \text{ (Region II)}$$

In regions I

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\alpha y & \alpha x & 0 \end{vmatrix}$$

$$= 2\alpha\hat{k}$$

In region II

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{\alpha r_o^2 y}{x^2 + y^2} & \dfrac{\alpha r_o^2 x}{x^2 + y^2} & 0 \end{vmatrix} = 0$$

21. If $\vec{r}$ is the position vector of the point (x, y, z) w.r.t. origin
Prove that

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

Find $f(r)$ such that $\nabla^2 f(r) = 0$

Solution.

$$\nabla^2 f(r) = \nabla \cdot \nabla f(r)$$

$$\nabla f(r) = \sum \hat{i} \frac{\partial}{\partial x} f(r)$$

$$= \sum \hat{i}\, f'(r) \frac{\partial r}{\partial x}$$

$$= \sum \hat{i}\, f'(r) \frac{x}{r}$$

$$= \frac{f'(r)}{r} \sum \hat{i}\, x$$

$$= \frac{f'(r)}{r} \vec{r}$$

$$\nabla^2 f(r) = \nabla \cdot \left(\frac{f'(r)}{r} \vec{r}\right)$$

$$= \sum \hat{i} \cdot \frac{\partial}{\partial x}\left(\frac{1}{r} f'(r)\vec{r}\right)$$

$$= \sum \hat{i} \cdot \left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right) f'(r)\vec{r} + \frac{1}{r}\frac{\partial}{\partial x}\big(f'(r)\big)\vec{r} + \frac{1}{r} f'(r) \frac{\partial \vec{r}}{\partial x}\right]$$

$$= \sum \hat{i} \cdot \left[-\frac{1}{r^2} \cdot \frac{x}{r} f'(r)\vec{r} + \frac{1}{r} f''(r) \frac{\partial r}{\partial x}\vec{r} + \frac{1}{r} f'(r)\hat{i}\right]$$

$$= \sum\left[-\frac{f'(r)}{r^3}\cdot x(\hat{i}.\bar{r})+\frac{1}{r}f''(r)\frac{x}{r}\cdot(\hat{i}.\bar{r})+\frac{1}{r}f'(r)\hat{i}\cdot\hat{i}\right]$$

$$= -\frac{f'(r)}{r^3}\Sigma x^2+\frac{f''(r)}{r^2}\Sigma x^2+\frac{1}{r}f'(r)\Sigma 1$$

$$= -\frac{f'(r)}{r}+f''(r)+\frac{3}{r}f'(r)$$

$$= f''(r)+\frac{2}{r}f'(r)$$

Now, let us find $f(r)$ such that $\nabla^2 f(r)=0$

Let $\quad g(r) = f'(r)$

Now, $\quad \nabla^2 f(r)=0$

$$\Rightarrow \quad f''(r)+\frac{2}{r}f'(r)=0$$

$$\Rightarrow \quad g'(r)+\frac{2}{r}g(r)=0$$

$$\Rightarrow \quad \frac{dg}{dr}+\frac{2}{r}g=0$$

$$\Rightarrow \quad \frac{dg}{g}+2\frac{dr}{r}=0$$

Integrating

$$\int\frac{dg}{g}+2\int\frac{dr}{r}=\text{const}$$

$$\Rightarrow \quad gr^2=C_1$$

$$\because \quad g(r)=\frac{C_1}{r^2}$$

$$\frac{df}{dr}=\frac{C_1}{r^2}$$

$$f=\int\frac{C_1}{r^2}dr+C_2$$

$$f(r)=-\frac{C_1}{r}+C_2$$

22. Evaluate *div* $\{\bar{a}\times(\bar{r}\times\bar{a})\}$ where $\bar{a}$ is a constant vector.

Solution.

$$\nabla\cdot\{\bar{a}\times(\bar{r}\times\bar{a})\}=\sum\hat{i}\cdot\frac{\partial}{\partial x}\{\bar{a}\times(\bar{r}\times\bar{a})\}$$

$$=\sum\hat{i}\cdot\left\{\bar{a}\times\left(\frac{\partial\bar{r}}{\partial x}\times\bar{a}\right)\right\}$$

$$=\sum\hat{i}\cdot\{\bar{a}\times(\hat{i}\times\bar{a})\}$$

$$=\sum\hat{i}\cdot\{(\bar{a}\cdot\bar{a})\hat{i}-(\bar{a}\cdot\hat{i})\bar{a}\}$$

$$=\sum\left[(\bar{a}\cdot\bar{a})(\hat{i}\cdot\hat{i})-(\bar{a}\cdot\hat{i})(\bar{a}\cdot\hat{i})\right]$$

$$=a^2\Sigma 1-\Sigma a_x^2$$

$$=3a^2-a^2=2a^2$$

23. If $\bar{f} = \frac{1}{r}\bar{r}$; find grad $(div\,\bar{f})$.

Solution.

$$div\,\bar{f} = \sum \hat{i}\cdot\frac{\partial}{\partial x}\left(\frac{1}{r}\bar{r}\right)$$

$$= \sum \hat{i}\cdot\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\bar{r} + \frac{1}{r}\frac{\partial \bar{r}}{\partial x}\right)$$

$$= \sum \hat{i}\cdot\left(-\frac{1}{r^2}\cdot\frac{x}{r}\bar{r} + \frac{1}{r}\hat{i}\right)$$

$$= -\frac{1}{r^3}\sum x(\hat{i}\cdot\bar{r}) + \sum\frac{1}{r}(\hat{i}\cdot\hat{i})$$

$$= -\frac{1}{r^3}\sum x^2 + \frac{1}{r}\sum 1$$

$$= -\frac{1}{r} + \frac{3}{r} = \frac{2}{r}$$

$$\text{grad}\,(div\,\bar{f}) = \text{grad}\left(\frac{2}{r}\right)$$

$$= \sum \hat{i}\frac{\partial}{\partial x}\left(\frac{2}{r}\right)$$

$$= \sum \hat{i}\left(-\frac{2}{r^2}\right)\frac{\partial r}{\partial x}$$

$$= -\frac{2}{r^3}\cdot\sum x\hat{i}$$

$$= -\frac{2}{r^3}\cdot\bar{r}$$

24. Prove that curl $(\bar{a}\times\bar{r})\,r^n = (n+2)r^n\bar{a} - nr^{n-2}(\bar{r}\cdot\bar{a})\bar{r}$

Solution.

$$\text{curl}\,(\bar{a}\times\bar{r})r^n = \sum \hat{i}\times\frac{\partial}{\partial x}\left[(\bar{a}\times\bar{r})r^n\right]$$

$$= \sum \hat{i}\times\left[\left(\bar{a}\times\frac{\partial\bar{r}}{\partial x}\right)r^n + (\bar{a}\times\bar{r})\frac{\partial}{\partial x}(r^n)\right]$$

$$= \sum \hat{i}\times\left[(\bar{a}\times\hat{i})r^n + (\bar{a}\times\bar{r})nr^{n-1}\frac{\partial r}{\partial x}\right]$$

$$= \sum \hat{i}\times(\bar{a}\times\hat{i})r^n + \sum \hat{i}\times(\bar{a}\times\bar{r})nr^{n-1}\frac{x}{r}$$

$$= r^n\Sigma\hat{i}\times(\bar{a}\times\hat{i}) + r^{n-2}\Sigma x\hat{i}\times(\bar{a}\times\bar{r})$$

$$= r^n\Sigma\left[(\hat{i}\cdot\hat{i})\bar{a} - (\hat{i}\cdot\bar{a})\hat{i}\right] + nr^{n-2}\left[\bar{r}\times(\bar{a}\times\bar{r})\right]$$

$$= r^n\left[\bar{a}\Sigma 1 - \Sigma(\hat{i}\cdot\bar{a})\hat{i}\right] + nr^{n-2}\left[(\bar{r}\cdot\bar{r})\bar{a} - (\bar{r}\cdot\bar{a})\bar{r}\right]$$

$$= 3r^n\bar{a} - r^n\bar{a} + nr^n\bar{a} - nr^{n-2}(\bar{r}\cdot\bar{a})\bar{r}$$

$$= (n+2)r^n\bar{a} - nr^{n-2}(\bar{r}\cdot\bar{a})\bar{r}$$

25. Prove that $\operatorname{div}\left\{\frac{f(r)}{r}\bar{r}\right\}=\frac{1}{r^2}\frac{d}{dr}(r^2 f)$

Solution.

$$\operatorname{div}\left\{\frac{f(r)}{r}\bar{r}\right\}=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left\{\frac{1}{r}f(r)\bar{r}\right\}$$

$$=\sum\hat{i}\cdot\left\{\frac{\partial}{\partial x}\left(\frac{1}{r}\right)f(r)\bar{r}+\frac{1}{r}\frac{\partial}{\partial x}(f(r))\bar{r}+\frac{1}{r}f(r)\frac{\partial\bar{r}}{\partial x}\right\}$$

$$=\sum\hat{i}\cdot\left\{-\frac{1}{r^2}\cdot\frac{x}{r}f(r)\bar{r}+\frac{1}{r}f'(r)\frac{x}{r}\cdot\bar{r}+\frac{1}{r}f(r)\hat{i}\right\}$$

$$=\sum\left\{-\frac{f(r)}{r^3}x(\hat{i}\cdot\bar{r})+\frac{1}{r^2}f'(r)x(\hat{i}\cdot\bar{r})+\frac{1}{r}f(r)(\hat{i}\cdot\hat{i})\right\}$$

$$=-\frac{f(r)}{r^3}\Sigma x^2+\frac{1}{r^2}f'(r)\Sigma x^2+\frac{1}{r}f(r)\Sigma 1$$

$$=-\frac{f(r)}{r}+f'(r)+3\frac{f(r)}{r}$$

$$=f'(r)+2\frac{f(r)}{r}$$

$$=\frac{1}{r^2}\left[r^2 f'(r)+2r\,f(r)\right]$$

$$=\frac{1}{r^2}\frac{d}{dr}(r^2 f)$$

26. Prove that $\nabla.\left\{r\nabla\left(\frac{1}{r^3}\right)\right\}=\frac{3}{r^4}$

Solution.

$$\nabla\left(\frac{1}{r^3}\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)$$

$$=\sum\hat{i}\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}\right)$$

$$=\sum\hat{i}\left(-\frac{3x}{r^5}\right)$$

$$=-\frac{3}{r^5}\Sigma\hat{i}x$$

$$=-\frac{3\bar{r}}{r^5}$$

$$r\nabla\left(\frac{1}{r^3}\right)=-\frac{3\bar{r}}{r^4}$$

$$\nabla\cdot\left(r\nabla\left(\frac{1}{r^3}\right)\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(-3\frac{\bar{r}}{r^4}\right)$$

$$=-3\sum\hat{i}\cdot\left\{\frac{\partial}{\partial x}\left(\frac{1}{r^4}\right)\bar{r}+\frac{1}{r^4}\cdot\frac{\partial\bar{r}}{\partial x}\right\}$$

$$=-3\sum\left\{-\frac{4}{r^5}\cdot\frac{\partial r}{\partial x}(\hat{i}\cdot\bar{r})+\frac{1}{r^4}(\hat{i}\cdot\hat{i})\right\}$$

$$= -3\sum\left\{-\frac{4}{r^6}\cdot x^2+\frac{1}{r^4}\right\}$$

$$= -3\left\{-\frac{4}{r^6}\Sigma x^2+\frac{1}{r^4}\Sigma 1\right\}$$

$$= -3\left\{-\frac{4}{r^4}+\frac{3}{r^4}\right\}=\frac{3}{r^4}$$

27. Prove that

$$\vec{b}\cdot\nabla\left(\vec{a}.\nabla\left(\frac{1}{r}\right)\right)=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{\vec{a}\cdot\vec{b}}{r^3}$$

where $\vec{a}$ and $\vec{b}$ are constant vectors

Solution.

$$\nabla\left(\frac{1}{r}\right)=\sum\hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$=\sum\hat{i}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}\right)$$

$$=\sum\hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right)$$

$$=-\frac{1}{r^2}\Sigma\hat{i}x$$

$$=-\frac{\vec{r}}{r^3}$$

$$\vec{a}\cdot\nabla\left(\frac{1}{r}\right)=-\frac{(\vec{a}\cdot\vec{r})}{r^3}$$

$$\nabla\left(\vec{a}\cdot\nabla\left(\frac{1}{r}\right)\right)=\sum\hat{i}\,\frac{\partial}{\partial x}\left(-\frac{(\vec{a}\cdot\vec{r})}{r^3}\right)$$

$$=-\sum\hat{i}\left(\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)(\vec{a}\cdot\vec{r})+\frac{1}{r^3}\left(\vec{a}\cdot\frac{\partial\vec{r}}{\partial x}\right)\right)$$

$$=-\sum\hat{i}\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}(\vec{a}\cdot\vec{r})+\frac{1}{r^3}(\vec{a}\cdot\hat{i})\right)$$

$$=\frac{3}{r^5}(\vec{a}\cdot\vec{r})(\Sigma\hat{i}x)-\frac{1}{r^3}\Sigma(\vec{a}\cdot\hat{i})\hat{i}$$

$$=\frac{3}{r^5}(\vec{a}\cdot\vec{r})\vec{r}-\frac{1}{r^3}\vec{a}$$

$$\vec{b}\cdot\nabla\left(\vec{a}.\nabla\left(\frac{1}{r}\right)\right)=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{1}{r^3}(\vec{a}\cdot\vec{b})$$

28. If $\vec{a}$ is a constant vector, prove that

$$\operatorname{curl}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=-\frac{\vec{a}}{r^3}+\frac{3\vec{r}}{r^5}(\vec{a}\cdot\vec{r})$$

Solution.

$$\operatorname{curl}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=\sum\hat{i}\times\frac{\partial}{\partial x}\left[(\vec{a}\times\vec{r})\frac{1}{r^3}\right]$$

$$= \sum \hat{i} \times \left[\left(\vec{a} \times \frac{\partial \vec{r}}{\partial x}\right)\frac{1}{r^3} + (\vec{a} \times \vec{r})\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)\right]$$

$$= \sum \hat{i} \times \left[(\vec{a} \times \hat{i})\frac{1}{r^3} + (\vec{a} \times \vec{r})\left(-\frac{3}{r^4} \cdot \frac{\partial r}{\partial x}\right)\right]$$

$$= \frac{1}{r^3}\Sigma \hat{i} \times (\vec{a} \times \hat{i}) - \frac{3}{r^5}\Sigma \hat{i}x \times (\vec{a} \times \vec{r})$$

$$= \frac{1}{r^3}\left(\Sigma(\hat{i} \cdot \hat{i})\vec{a} - \Sigma(\hat{i} \cdot \vec{a})\hat{i}\right) - \frac{3}{r^5} \cdot \vec{r} \times (\vec{a} \times \vec{r})$$

$$= \frac{1}{r^3}(3\vec{a} - \vec{a}) - \frac{3}{r^5}[(\vec{r} \cdot \vec{r})\vec{a} - (\vec{a} \cdot \vec{r})\vec{r}]$$

$$= \frac{2\vec{a}}{r^3} - \frac{3}{r^5}\left[r^2\vec{a} - (\vec{a} \cdot \vec{r})\vec{r}\right]$$

$$= -\frac{\vec{a}}{r^3} + \frac{3}{r^5}(\vec{a} \cdot \vec{r})\vec{r}$$

29. Evaluate $\nabla^2\left[\nabla \cdot \left(\frac{\vec{r}}{r^2}\right)\right]$

Solution.

$$\nabla \cdot \left(\frac{\vec{r}}{r^2}\right) = \sum \hat{i} \cdot \frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^2}\right)$$

$$= \sum \hat{i} \cdot \left[\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right)\vec{r} + \frac{1}{r^2}\frac{\partial \vec{r}}{\partial x}\right]$$

$$= \sum \hat{i} \cdot \left[-\frac{2}{r^3} \cdot \frac{x\vec{r}}{r} + \frac{1}{r^2}\hat{i}\right]$$

$$= \sum \left[-\frac{2x}{r^4} \cdot (\hat{i} \cdot \vec{r}) + \frac{1}{r^2}\hat{i} \cdot \hat{i}\right]$$

$$= \sum \left[-\frac{2x^2}{r^4} + \frac{1}{r^2}\right]$$

$$= -\frac{2}{r^4}\Sigma x^2 + \frac{1}{r^2}\Sigma 1$$

$$= -\frac{2}{r^2} + \frac{3}{r^2} = \frac{1}{r^2}$$

Now, $$\nabla^2\left[\nabla \cdot \left(\frac{\vec{r}}{r^2}\right)\right] = \nabla^2\left(\frac{1}{r^2}\right)$$

$$= \nabla \cdot \nabla\left(\frac{1}{r^2}\right)$$

$$\nabla\left(\frac{1}{r^2}\right) = \sum \hat{i} \frac{\partial}{\partial x}\left(\frac{1}{r^2}\right)$$

$$= \sum \hat{i}\left(-\frac{2}{r^3} \cdot \frac{\partial r}{\partial x}\right)$$

$$= -\frac{2}{r^4}\sum x\hat{i}$$

$$= -\frac{2\vec{r}}{r^4}$$

$$\nabla\cdot\nabla\left(\frac{1}{r^2}\right) = \nabla\cdot\left(-\frac{2\vec{r}}{r^4}\right)$$

$$= \sum \hat{i}\cdot\frac{\partial}{\partial x}\left(-\frac{2\vec{r}}{r^4}\right)$$

$$= -2\sum \hat{i}\cdot\left[\frac{\partial}{\partial x}\left(\frac{1}{r^4}\right)\vec{r} + \frac{1}{r^4}\frac{\partial\vec{r}}{\partial x}\right]$$

$$= -2\sum \hat{i}\cdot\left[-\frac{4}{r^5}\frac{\partial r}{\partial x}\vec{r} + \frac{1}{r^4}\hat{i}\right]$$

$$= -2\sum\left[-\frac{4x}{r^6}\cdot(\hat{i}\cdot\vec{r}) + \frac{1}{r^4}\hat{i}\cdot\hat{i}\right]$$

$$= -2\sum\left[-\frac{4x^2}{r^6} + \frac{1}{r^4}\right]$$

$$= -2\left[-\frac{4}{r^6}\Sigma x^2 + \frac{1}{r^4}\Sigma 1\right]$$

$$= -2\left[-\frac{4}{r^4} + \frac{3}{r^4}\right] = \frac{2}{r^4}$$

30. Expand

(i) $\nabla\left(\vec{a}\cdot\frac{\vec{r}}{r^n}\right)$ (ii) $\nabla\cdot\left(\vec{a}\times\frac{\vec{r}}{r^n}\right)$ (ii) $\nabla\times\left(\vec{a}\times\frac{\vec{r}}{r^n}\right)$

where $\vec{a}$ is a constant vector

Solution.

(i) $$\nabla\left(\vec{a}\cdot\frac{\vec{r}}{r^n}\right) = \sum \hat{i}\frac{\partial}{\partial x}\left(\vec{a}\cdot\frac{\vec{r}}{r^n}\right)$$

$$= \sum \hat{i}\left[\vec{a}\cdot\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)\right]$$

$$= \sum \hat{i}\left[\vec{a}\cdot\left(\frac{\partial}{\partial x}\left(\frac{1}{r^n}\right)\vec{r} + \frac{1}{r^n}\frac{\partial\vec{r}}{\partial x}\right)\right]$$

$$= \sum \hat{i}\left[\vec{a}\cdot\left(-\frac{n}{r^{n+1}}\cdot\frac{\partial r}{\partial x}\vec{r} + \frac{1}{r^n}\hat{i}\right)\right]$$

$$= \sum \hat{i}\left[-\frac{n}{r^{n+2}}x(\vec{a}\cdot\vec{r}) + \frac{1}{r^n}(\vec{a}\cdot\hat{i})\right]$$

$$= -\frac{n}{r^{n+2}}(\vec{a}\cdot\vec{r})(\Sigma\hat{i}x) + \frac{1}{r^n}\Sigma(\vec{a}\cdot\hat{i})\hat{i}$$

$$= -\frac{n(\vec{a}\cdot\vec{r})}{r^{n+2}}\vec{r} + \frac{1}{r^n}\vec{a}$$

(ii) $$\nabla\cdot\left(\vec{a}\times\frac{\vec{r}}{r^n}\right) = \sum \hat{i}\cdot\frac{\partial}{\partial x}\left(\vec{a}\times\frac{\vec{r}}{r^n}\right)$$

$$= \sum \hat{i}\cdot\left[\vec{a}\times\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)\right]$$

$$= \sum \hat{i}\cdot\left[\vec{a}\times\left(\frac{\partial}{\partial x}\left(\frac{1}{r^n}\right)\cdot\vec{r}+\frac{1}{r^n}\frac{\partial\vec{r}}{\partial x}\right)\right]$$

$$= \sum \hat{i}\cdot\left[\vec{a}\times\left(-\frac{n}{r^{n+1}}\cdot\frac{\partial r}{\partial x}\vec{r}+\frac{1}{r^n}\hat{i}\right)\right]$$

$$= \sum \hat{i}\cdot\left[\vec{a}\times\left(-\frac{nx}{r^{n+2}}\vec{r}+\frac{1}{r^n}\hat{i}\right)\right]$$

$$= \sum \hat{i}\cdot\left[\vec{a}\times\left(-\frac{nx}{r^{n+2}}\right)\vec{r}\right]+\sum \hat{i}\cdot\left(\vec{a}\times\frac{1}{r^n}\hat{i}\right)$$

$$= -\frac{n}{r^{n+2}}\Sigma\hat{i}x\cdot(\vec{a}\times\vec{r})+0$$

$$= -\frac{n}{r^{n+2}}\vec{r}\cdot(\vec{a}\times\vec{r})$$

$$= 0$$

(iii) $$\nabla\times\left(\vec{a}\times\frac{\vec{r}}{r^n}\right) = \sum \hat{i}\times\frac{\partial}{\partial x}\left(\vec{a}\times\frac{\vec{r}}{r^n}\right)$$

$$= \sum \hat{i}\times\left[\vec{a}\times\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)\right]$$

$$= \sum \hat{i}\times\left[\vec{a}\times\left(\frac{\partial}{\partial x}\left(\frac{1}{r^n}\right)\vec{r}+\frac{1}{r^n}\frac{\partial\vec{r}}{\partial x}\right)\right]$$

$$= \sum \hat{i}\times\left[\vec{a}\times\left(-\frac{n}{r^{n+1}}\cdot\frac{\partial r}{\partial x}\vec{r}+\frac{1}{r^n}\hat{i}\right)\right]$$

$$= \sum \hat{i}\times\left[\vec{a}\times\left(-\frac{n}{r^{n+2}}x\vec{r}+\frac{1}{r^n}\hat{i}\right)\right]$$

$$= \sum \hat{i}\times\left[\vec{a}\times\left(-\frac{n}{r^{n+2}}x\right)\vec{r}\right]+\sum \hat{i}\times\left(\vec{a}\times\frac{1}{r^n}\hat{i}\right)$$

$$= -\frac{n}{r^{n+2}}\sum \hat{i}x\times(\vec{a}\times\vec{r})+\frac{1}{r^n}\sum \hat{i}\times\left(\vec{a}\times\hat{i}\right)$$

$$= -\frac{n}{r^{n+2}}\vec{r}\times(\vec{a}\times\vec{r})+\frac{1}{r^n}\Sigma\left[(\hat{i}\cdot\hat{i})\vec{a}-(\hat{i}\cdot\vec{a})\hat{i}\right]$$

$$= -\frac{n}{r^{n+2}}\left[(\vec{r}\cdot\vec{r})\vec{a}-(\vec{r}\cdot\vec{a})\vec{r}\right]+\frac{1}{r^n}\left[\vec{a}\Sigma 1-\Sigma(\hat{i}\cdot\vec{a})\hat{i}\right]$$

$$= -\frac{n}{r^{n+2}}\left[(r^2\vec{a}-(\vec{r}\cdot\vec{a})\vec{r}\right]+\frac{1}{r^n}\left[3\vec{a}-\vec{a}\right]$$

$$= -\frac{n}{r^n}\vec{a}+n\frac{(\vec{r}\cdot\vec{a})}{r^{n+2}}\vec{r}+\frac{2\vec{a}}{r^n}$$

$$= \frac{(2-n)\vec{a}}{r^n}+n\frac{(\vec{r}\cdot\vec{a})}{r^{n+2}}\vec{r}$$

31. Show that

$$\text{curl}\left(\hat{k}\times\text{grad}\frac{1}{r}\right)+\text{grad}\left(\hat{k}\cdot\text{grad}\frac{1}{r}\right)=0$$

where r is the distance from the origin and $\hat{k}$ the unit vector in the direction OZ.

Solution.

$$\nabla\left(\frac{1}{r}\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$=-\frac{\vec{r}}{r^3}$$

$$\text{curl}\left[\hat{k}\times grad\left(\frac{1}{r}\right)\right]=\nabla\times\left[\left(\hat{k}\times\left(-\frac{\vec{r}}{r^3}\right)\right)\right]\left(\text{grad}\left(\frac{1}{r}\right)=-\frac{\vec{r}}{r^3}\right)$$

$$=-\sum\hat{i}\times\frac{\partial}{\partial x}\left[(\hat{k}\times\vec{r})\frac{1}{r^3}\right]$$

$$=-\sum\hat{i}\times\left[\left(\hat{k}\times\frac{\partial\vec{r}}{\partial x}\right)\frac{1}{r^3}+(\hat{k}\times\vec{r})\left(-\frac{3}{r^4}\cdot\frac{x}{r}\right)\right]$$

$$=-\frac{1}{r^3}\left[\hat{i}\times(\hat{k}\times\hat{i})+\hat{j}\times(\hat{k}\times\hat{j})+\hat{k}\times(\hat{k}\times\hat{k})\right]+\frac{3}{r^5}\left(\vec{r}\times(\hat{k}\times\vec{r})\right)$$

$$=-\frac{1}{r^3}[\hat{k}+\hat{k}+0]+\frac{3}{r^5}\left(\vec{r}\times(\hat{k}\times\vec{r})\right)$$

$$=-\frac{2}{r^3}\hat{k}+\frac{3}{r^5}[r^2\hat{k}-(\hat{k}\cdot\vec{r})\vec{r}]$$

$$=-\frac{2}{r^3}\hat{k}+\frac{3}{r^3}\hat{k}-\frac{3z}{r^5}\vec{r}$$

$$=\frac{\hat{k}}{r^3}-\frac{3z}{r^5}\vec{r}$$

$$\text{grad}\left(\hat{k}.\text{grad}\frac{1}{r}\right)=\nabla\left(\hat{k}.\left(-\frac{\vec{r}}{r^3}\right)\right)=-\sum\hat{i}\frac{\partial}{\partial x}\left((\hat{k}.\vec{r}).\frac{1}{r^3}\right)$$

$$=-\sum i\frac{\partial}{\partial x}\left(\frac{z}{r^3}\right)$$

$$=-\left[\frac{\hat{k}}{r^3}+z\sum\hat{i}\left(\frac{-3}{r^4}.\frac{x}{r}\right)\right]$$

$$=-\frac{\hat{k}}{r^3}+\frac{3z}{r^5}\vec{r}$$

So, $$\text{curl}\left[\hat{k}\times grad\left(\frac{1}{r}\right)\right]+grad\left(\hat{k}.grad\frac{1}{r}\right)=0$$

32. Evaluate $\nabla^2\left(\frac{x}{r^3}\right)$.

Solution.

$$\nabla^2\left(\frac{x}{r^3}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3}-\frac{3x}{r^4}\cdot\frac{\partial r}{\partial x}$$

$$= \frac{1}{r^3}-\frac{3x^2}{r^5}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right)$$

$$= -\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}-\frac{6x}{r^5}-3x^2\left(-\frac{5}{r^6}\right)\cdot\frac{\partial r}{\partial x}$$

$$= -\frac{3x}{r^5}-\frac{6x}{r^5}+\frac{15x^3}{r^7} = -\frac{9x}{r^5}+\frac{15x^3}{r^7}$$

Now,
$$\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right) = x\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial y}\right)$$

$$= -\frac{3xy}{r^5}$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial y}\left(-\frac{3xy}{r^5}\right)$$

$$= -\frac{3x}{r^5}-3xy\frac{\partial}{\partial y}\left(\frac{1}{r^5}\right)$$

$$= -\frac{3x}{r^5}-3xy\left(-\frac{5}{r^6}\cdot\frac{\partial r}{\partial y}\right)$$

$$= -\frac{3x}{r^5}+\frac{15xy^2}{r^7}$$

$$\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) = x\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial z}\right)$$

$$= -\frac{3xz}{r^5}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial z}\left(-\frac{3xz}{r^5}\right)$$

$$= -\frac{3x}{r^5}-3xz\frac{\partial}{\partial z}\left(\frac{1}{r^5}\right)$$

$$= -\frac{3x}{r^5}-3xz\left(-\frac{5}{r^6}\cdot\frac{\partial r}{\partial z}\right)$$

$$= -\frac{3x}{r^5}+\frac{15xz^2}{r^7}$$

So,
$$\nabla^2\left(\frac{x}{r^3}\right) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right)$$

$$= \left(-\frac{9x}{r^5}+\frac{15x^3}{r^7}\right)+\left(-\frac{3x}{r^5}+\frac{15xy^2}{r^7}\right)+\left(-\frac{3x}{r^5}+\frac{15xz^2}{r^7}\right)$$

$$= -\frac{15x}{r^5}+\frac{15x(x^2+y^2+z^2)}{r^7}$$

$$= -\frac{15x}{r^5}+\frac{15x}{r^5}=0$$

EXERCISE – 2

1. If $\vec{a}$ is constant vector, obtain
   (a) div $(\vec{r}\times\vec{a})$ **Ans.** 0
   (b) curl $(\vec{r}\times\vec{a})$ **Ans.** $-2\vec{a}$
2. Prove that $\nabla^2(r^n\vec{r}) = n(n-3)nr^{n-2}\vec{r}$.
3. Prove that divgrad$r^n = n(n+1)r^{n-2}$.
4. A vector function $\vec{f}$ is the product of a scalar function and the gradient of a scalar function, show that
   $$\vec{f}.\text{curl}\,\vec{f} = 0$$
5. Prove that div$(\vec{A}\times\vec{r}) = \vec{r}\,.\,\text{curl}\,\vec{A}$
6. Verify that curl grad $f = 0$ where
   $$f = x^3y^2 + 2x^2y + z^3$$
7. Prove that
   $$\text{curl}(\psi\times\phi) = \nabla\psi\times\nabla\phi = -\text{curl}(\phi\nabla\psi)$$
8. Show that curl$(\vec{a}.\vec{r})\vec{a} = 0$ where $\vec{a}$ is a constant vector.
9. Prove that $\vec{a}.\{\nabla(\vec{f}.\vec{a}) - \nabla(\vec{f}\times\vec{a})\} = \text{div}\vec{f}$, where $\vec{a}$ is a constant unit vector.
10. If $\vec{r}$ is a the position vector of the point (x, y, z). Show that curl $(r^n\vec{r}) = 0$, where r is the modulus of $\vec{r}$.
11. If $\vec{f} = \frac{\vec{r}}{r}$, show that $\nabla\times\vec{f} = 0$.
12. If $\nabla^2 f(r) = 0$, show that $f(r) = c_1 \log r + c_2$ where $r^2 = x^2 + y^2$ and c_1 & c_2 are arbitrary constants.

Srinivasa Ramanujan

Srinivasa Ramanujan (1887-1920) hailed as an all-time great mathematician like Euler, Gauss or Jacobi, for his *natural genius*, has left behind 4000 original theorems, despite his lack of formal education and a short life-span. Ramanujan was awarded in 1916 the B.A. Degree by research of the Cambridge University. He was elected a Fellow of the Royal Society of London in Feb. 1918 being a "Research student in Mathematics Distinguished as a pure mathematician particularly for his investigations in elliptic functions and the theory of numbers" and he was elected to a Trinity College Fellowship in Oct 1918.

During his five year stay in Cambridge he published 21 papers, five of which were in collaboration with Prof. G.H. Hardy and these as well as his earlier publications before he set sail to England are all contained in the "Collected Papers of Srinivasa Ramanujan".

4

Line Integral

BASIC CONCEPTS

Oriented Curve : Let C be a curve in space from A to B.

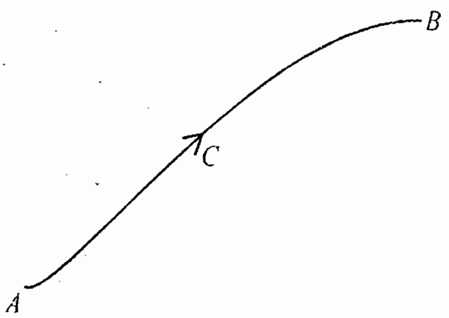


Figure 4.1

It can be oriented by taking one of the possible two direction as positive. If the direction is from A to B, is taken as positive then the direction from B to A will be negative. If the initial point A and terminal point B coincide then the curve C is called closed curve. (Figure 4.2)

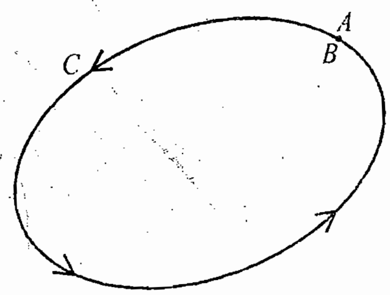


Figure 4.2

Smooth Curve : Let the curve C is represented by parametric equation.

$$x = x(t); \quad y = y(t); \quad z = z(t)$$

Then each point on the curve C is represented by position vector.

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

A curve is said to be smooth if the function $\vec{r}(t)$ is continuous and has continuous first derivatives not equal to zero for all values of t. There is unique tangent at each point on the curve C.

Piecewise Smooth Curve: A curve C is said to be piecewise smooth if it is composed of finite number of smooth curves. For example:–

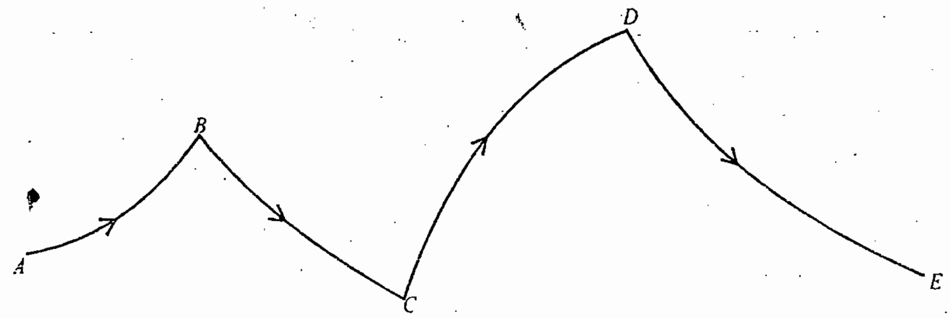


Figure 4.3

The above curve (Figure 4.3) is piecewise smooth curve because it consists of four pieces of smooth curves, AB, BC, CD and DE.

The rectangle consists of four pieces of smooth curves.

Line Integrals: The integral which is carried on a curve is called line integral.

Mathematically, line integral is written as

$$\int_c \vec{F}.d\vec{r}$$

On the curve C, only one variable is independent, other two variables are dependent.

If $\qquad \vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

Then, $\vec{F}.d\vec{r} = f_1 dx + f_2 dy + f_3 dz$

where f_1, f_2, f_3 are functions of x, y, z.

If we take x as independent variables.

then y and z can be expressed in terms of x using the equation of curve

$$y = g(x) \Rightarrow dy = g'(x)\, dx$$
$$z = h(x) \Rightarrow dz = h'(x)\, dx$$

Now, $\vec{F}.d\vec{r} = f_1 dx + f_2 dy + f_3 dz$ will be reduced to $f(x)\,dx$.

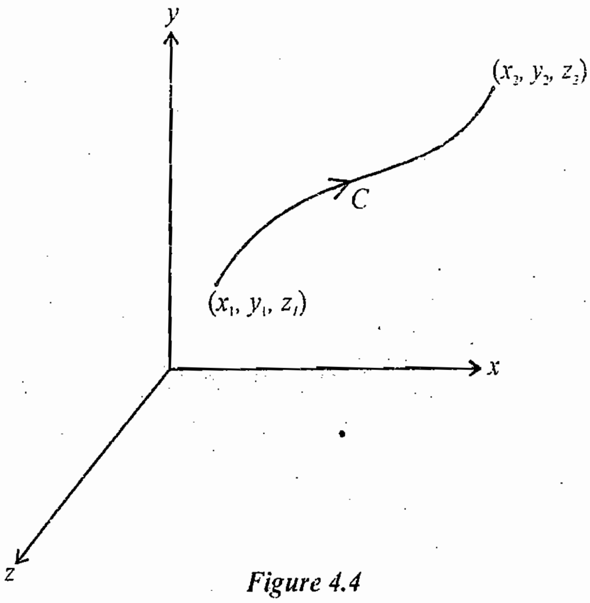


Figure 4.4

So, the line integral $\int_c \vec{F}.d\vec{r}$ reduces to a simple definite integral.

$$\int_c \vec{F}.d\vec{r} = \int_{x_1}^{x_2} f(x)dx$$

which can be solved using usual techniques.

Similarly, if y is taken as independent variable then x and z can be expressed in terms of y using equation of curve

$$x = g(y) \Rightarrow dx = g'(y)\, dy$$
$$z = h(y) \Rightarrow dz = h'(y)\, dy$$

So, $\quad \vec{F}.d\vec{r} = f_1 dx + f_2 dy + f_3 dz$ will be reduced to $f(y)\, dy$.

So, line integral $\int_c \vec{F}.d\vec{r}$ reduces to a simple definite integral $\int_{y_1}^{y_2} f(y)dy$ which can be solved using usual techniques.

Similarly, if z is taken as independent variable, then x and y can be expressed in terms of z using equation of curve.

$$x = g(z) \Rightarrow dx = g'(z)\,dz$$
$$y = h(z) \Rightarrow dy = h'(z)\,dz$$

So, $\vec{F}.d\vec{r} = f_1dx + f_2dy + f_3dz$ will be reduced to $f(z)\,dz$.

So, line integral $\int_c \vec{F}.d\vec{r}$ reduces to a simple definite integral $\int_{z_1}^{z_2} f(z)dz$ which can be solved using usual techniques.

Sometimes the equation of curve is given in terms of parameters, x, y, z are expressed in terms of parameters t using parametric equation of curve

$$x = g_1(t) \Rightarrow dx = g_1'(t)\,dt$$
$$y = g_2(t) \Rightarrow dy = g_2'(t)\,dt$$
$$z = g_3(t) \Rightarrow dz = g_3'(t)\,dt$$

So, $\vec{F}.d\vec{r} = f_1dx + f_2dy + f_3dz$ will be reduced to $f(t)\,dt$.

So, line integral $\int_c \vec{F}.d\vec{r}$ reduces to a simple definite integral $\int_{t_1}^{t_2} f(t)dt$ which can be solved using usual techniques.

So, there are the four ways of solving line integral by converting it into a definite integral.

SOLVED EXAMPLES

1. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$ determine the value of $\int_c \vec{F}.d\vec{r}$ where C is the curve $y = 2x^2$ in the xy plane from $(0, 0)$ to $(1, 2)$.

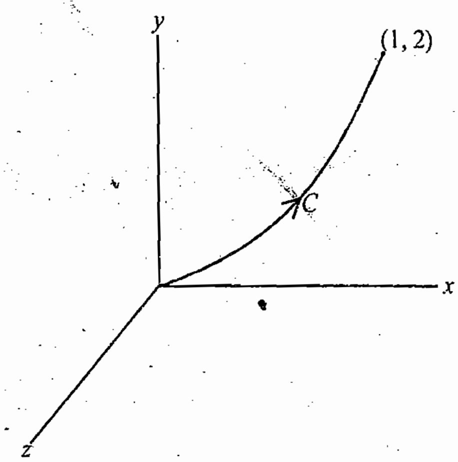


Figure 4.5

Solution.

The curve lies in xy plane, so, $z = 0$. z can never be taken as independent variable z is a dependent variable. Now, out of x and y, any one variable can be taken as independent.

Suppose x is taken as independent variable

$$y = 2x^2,\ dy = 4xdx$$

$$\vec{F}.d\vec{r} = 3xydx - y^2dy$$
$$= 6x^3dx - 4x^4.\ 4xdx$$
$$= (6x^3 - 16x^5)dx$$

So, the line integral $\int_C \vec{f}.d\vec{r}$ reduces to a definite integral.

$$\int_0^1 (6x^3 - 16x^5)dx$$

$$= 6\frac{x^4}{4}\Big|_0^1 - 16\frac{x^6}{6}\Big|_0^1$$

$$= -\frac{7}{6}$$

If y is taken as independent variable then x can be expressed in terms of y as

$$x = \sqrt{\frac{y}{2}}$$

$$dx = \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{y}}dy$$

So, $\vec{f}.d\vec{r} = 3xydx - y^2dy$

$$= 3y\sqrt{\frac{y}{2}}.\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{y}}dy - y^2dy$$

$$= \left(\frac{3}{4}y - y^2\right)dy$$

So, the line integral $\int \vec{f}.d\vec{r}$ reduces to a definite integral

$$\int_0^2 \left(\frac{3}{4}y - y^2\right)dy$$

$$= \frac{3}{8}y^2 - \frac{y^3}{3}\Big|_0^2$$

$$= -\frac{7}{6}$$

2. Evaluate $\int_c \vec{F}.d\vec{r}$ where $\vec{F} = (x^2 - y^2)\hat{i} + xy\hat{j}$ and curve C is arc of the curve $y = x^2$ from (0, 0) to (2, 4).

Solution.

$$\vec{F}.d\vec{r} = (x^2 - y^2)\,dx + xydy$$

Taking x as independent variable

$$y = x^2,\ dy = 2xdx$$

$$\vec{F}.d\vec{r} = (x^2 - y^2)\,dx + xydy$$

$$= (x^2 - x^4)\,dx + x^3.2xdx$$

$$= (x^2 + x^4)\,dx$$

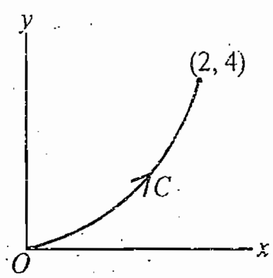


Figure 4.6

So, the line integral $\int \vec{F}.d\vec{r}$ reduces to a definite integral

$$\int_0^2 (x^2 + x^4)dx = \frac{x^3}{3} + \frac{x^5}{5}\Big|_0^2 = \frac{136}{15}$$

WORK DONE BY A FORCE

Suppose a force acts on a particle and particles is displaced along a given path C. Then work done by the force $\vec{F}$ is given by line integral.

$$W = \int_C \vec{F}.d\vec{r}$$

The integration is being carried in the sense of displacement.

3. Find the work done when a force

$$\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$$

moves a particle in xy plane from (0, 0) to (1, 1) along the parabola $y^2 = x$

Solution.

Here, on the curve C, y can be taken as independent variable and

$x = y^2$, $dx = 2ydy$

workdone in moving a particle by displacement $d\vec{r}$

$$dW = \vec{F}.d\vec{r}$$
$$= (x^2 - y^2 + x)dx - (2xy + y)dy$$
$$= (y^4 - y^2 + y^2).2ydy - (2y^2 . y + y)\,dy$$
$$= (2y^5 - 2y^3 - y)\,dy$$

Hence, work done is moving a particle from O to P is given by

$$W = \int_0^1 (2y^5 - 2y^3 - y)\,dy = 2\frac{y^6}{6} - 2\frac{y^4}{4} - \frac{y^2}{2}\Bigg|_0^1 = -\frac{2}{3}$$

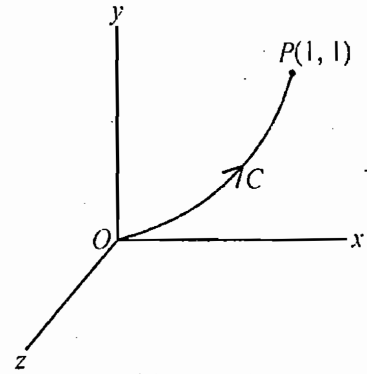


Figure 4.7

4. Evaluate $\oint xdy - ydx$ around a circle $x^2 + y^2 = r^2$

Solution.

Let C denotes the circle. The parametric equations of circle is

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Here, x and y have been expressed in terms of parameter which varies from 0 to 2π as one traverses the circle.

$$x = r\cos\theta \Rightarrow dx = -r\sin\theta\, d\theta$$
$$y = r\sin\theta \Rightarrow dy = r\cos\theta\, d\theta$$
$$x\,dy - y\,dx = r\cos\theta\; r\cos\theta\, d\theta - r\sin\theta(-r\sin\theta)\,d\theta$$
$$= r^2 d\theta$$

So, $\displaystyle\oint_c xdy - ydx = r^2 \oint d\theta$

$$= 2\pi r^2$$

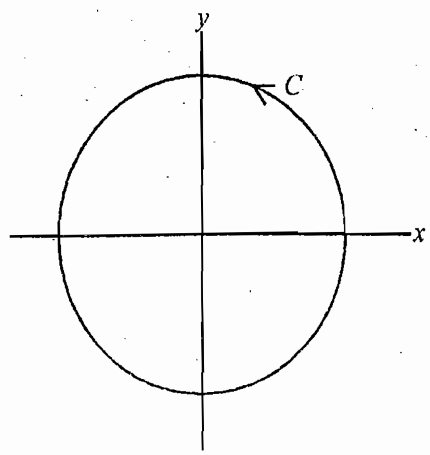


Figure 4.8

Here, r is a constant, because integral is carried over a circle.

5. Calculate the work done when a force $\vec{F} = xy\hat{i} + (x^2 + y^2)\hat{j}$ moves a particle in xy plane from (1, 0) to (3, 8) along the curve C, $y = x^2 - 1$.

Solution.

The curve C is $y = x^2 - 1$. Since, this is quadratic in x and linear in y with no xy terms. This is a parabola. Let us put this parabola in the form

$$(x-\alpha)^2 = 4a(y-\beta)$$
$$C:(x-0)^2 = (y+1)$$

This is a parabola with vertex at $(0,-1)$ and axis parallel to y axis.

On curve C, let us take x as indepedent variable. The dependent variable y can be written in terms of x as

$$y = x^2 - 1$$
$$dy = 2xdx$$

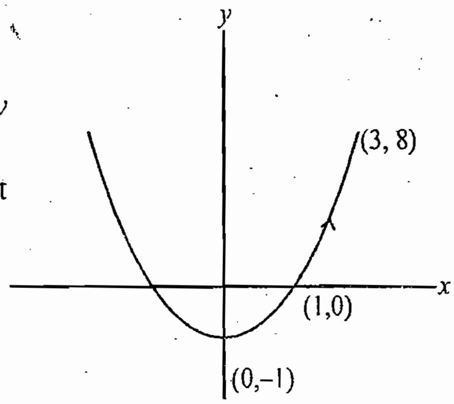


Figure 4.9

work done is moving a particle by displacement $d\vec{r}$

$$dW = \vec{F}.d\vec{r}$$
$$= xydx + (x^2 + y^2)dy$$
$$= x(x^2 - 1)dx + (x^2 + (x^2-1)^2)2xdx$$
$$= (2x^5 - x^3 + x)dx$$

So, work done is moving a particle from $(1, 0)$ to $(3, 8)$ along a curve C.

$$W = \int_c \vec{F}.d\vec{r} = \int_1^3 (2x^5 - x^3 + x)dx$$

$$= \left(2.\frac{x^6}{6} - \frac{x^4}{4} + \frac{x^2}{2}\right)\Bigg|_1^3 = 227$$

6. Evaluate the line integral $\int_c \vec{F}.d\vec{r}$ where $\vec{F} = (x+2y)\,\hat{i} + (2y-x)\,\hat{j}$ and C is curve in xy plane consisting of the straight lines from $(0, 0)$ to $(1, 0)$ and then to $(3, 4)$.

Solution.

The curve C consists of two pieces of smooth curves C_1 and C_2.

C_1 is straight line from $(0, 0)$ to $(1, 0)$ ie. $y = 0$

C_2 is straight line from $(1, 0)$ to $(3, 4)$

i.e. $$y - 0 = \left(\frac{4-0}{3-1}\right)\cdot(x-1)$$

or, $$y = 2x - 2$$

So, along C_1, $y = 0$, $dy = 0$ (x is an independent variable)

$$\vec{F}.d\vec{r} = x\,dx$$

Along C_2; $y = 2x-2$, $dy = 2dx$ (let us take x as independent variables).

$$\vec{F}.d\vec{r} = (x + 2y)\,dx + (2y - x)\,dy$$

on C_2: $$\vec{F}.d\vec{r} = (x + 2(2x-2))dx + (2(2x-2) - x).2dx$$
$$= (11x - 12)dx$$

So, $$\int_C \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r}$$

$$= \int_0^1 xdx + \int_1^3 (11x - 12)dx$$

$$= \frac{x^2}{2}\Bigg|_0^1 + \left(\frac{11}{2}x^2 - 12x\right)\Bigg|_1^3$$

$$= 20.5$$

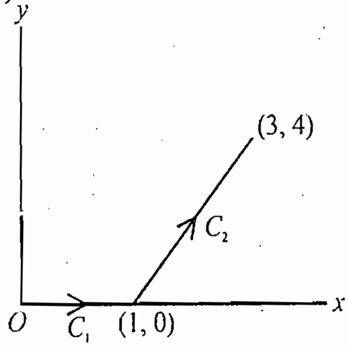


Figure 4.10

7. Evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$, where curve C is a rectangle in the xy plane bounded by $y = 0$, $x = a$, $y = b$, $x = 0$.

Solution.

The curve C as shown in figure 4.11 consists of four pieces of smooth curves C_1, C_2, C_3 & C_4.

$$\vec{F}.d\vec{r} = (x^2 + y^2)dx - 2xydy$$

On C_1, $y = 0$, $dy = 0$, $\vec{F}.d\vec{r} = x^2dx$

On C_2, $x = a$, $dx = 0$, $\vec{F}.d\vec{r} = -2aydy$

On C_3, $y = b$, $dy = 0$, $\vec{F}.d\vec{r} = (x^2 + b^2)dx$

On C_4, $x = 0$, $dx = 0$, $\vec{F}.d\vec{r} = 0$

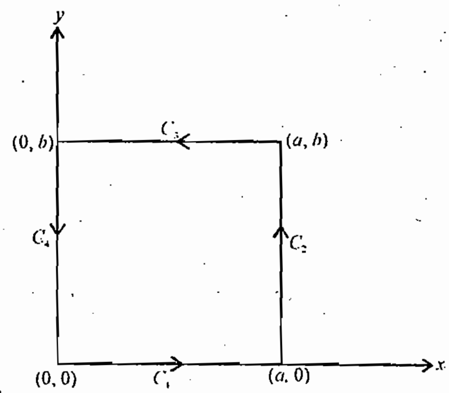


Figure 4.11

$$\oint \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r} + \int_{C_3} \vec{F}.d\vec{r} + \int_{C_4} \vec{F}.d\vec{r}$$

$$= \int_0^a x^2dx + \int_0^b -2aydy + \int_a^0 (x^2 + b^2)dx + \int_b^0 0.dy$$

$$= \left.\frac{x^3}{3}\right|_0^a + \left[-ay^2\right]_0^b + \left[\frac{x^3}{3} + b^2x\right]_a^0$$

$$= \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$= -2ab^2$$

8. Find the total work done in moving a particles in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

Solution.

On curve C, the coordinates x, y, z are expressed in terms of parameter t.

$x = t^2 + 1$, $dx = 2tdt$

$y = 2t^2$, $dy = 4tdt$

$z = t^3$, $dz = 3t^2dt$

t varies from $t = 1$ to $t = 2$.

$$\vec{F}.d\vec{r} = 3xydx - 5zdy + 10xdz$$

$$= 3(t^2+1).2t^2.2tdt - 5t^3 4tdt + 10(t^2+1).3t^2dt$$

$$= (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$$

So, total work done,

$$W = \int_C \vec{F}.d\vec{r}$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)dt$$

$$= \left.\left(12\frac{t^6}{6} + 10\frac{t^5}{5} + 12\frac{t^4}{4} + 30\frac{t^3}{3}\right)\right|_1^2$$

$$= 303$$

9. Find the work done in moving a particle once around a circle C in the xy plane if the circle has a centre at the origin and radius 2 and if the force field $\vec{F}$ is given by

$$\vec{F} = (2x - y + 2z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}$$

Solution.

Equation of circle as shown in figure 4.12 is written in parametric form as

$x = 2\cos\theta \Rightarrow dx = -2\sin\theta\, d\theta$
$y = 2\sin\theta \Rightarrow dy = 2\cos\theta\, d\theta$
$z = 0 \Rightarrow dz = 0$

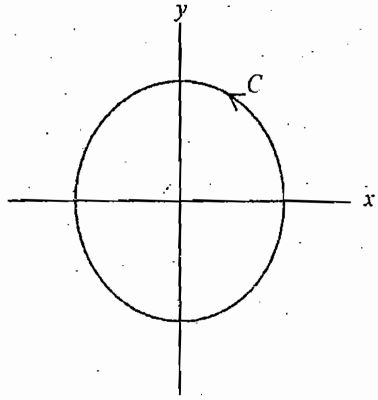


Figure 4.12

x, y, z are expressed in terms of parameter θ.

$$\vec{F}.d\vec{r} = (2x - y + 2z)dx + (x + y - z)dy + (3x - 2y - 5z)dz$$
$$= (4\cos\theta - 2\sin\theta)(-2\sin\theta)d\theta$$
$$+ (2\cos\theta + 2\sin\theta)(2\cos\theta)d\theta$$
$$+ (6\cos\theta - 4\sin\theta).0$$
$$= (4 - 4\sin\theta\cos\theta)\, d\theta$$

θ varies from 0 to 2π

So, total work done

$$W = \int_0^{2\pi} (4 - 4\sin\theta\cos\theta)d\theta$$
$$= 4\theta - 2\sin^2\theta\Big|_0^{2\pi}$$
$$= 8\pi$$

10. If $\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$. Evaluate $\int \vec{F}.d\vec{r}$ where C is a straight line joining (0, 0, 0) to (1, 1, 1).

Solution.

Equation of straight line joining (0, 0, 0) to (1, 1, 1) is given by $\dfrac{x-0}{1-0} = \dfrac{y-0}{1-0} = \dfrac{z-0}{1-0} = t$, where t is parameter.

In parametric form, equation of curve is given by

$x = t \Rightarrow dx = dt$
$y = t \Rightarrow dy = dt$
$z = t \Rightarrow dz = dt$

t varies from 0 to 1.

$$\vec{F}.d\vec{r} = (3x^2 + 6y)dx - 14yzdy + 20xz^2dz$$
$$= (20t^3 - 11t^2 + 6t)\, dt$$
$$\int_C \vec{F}.d\vec{r} = \int_0^1 (20t^3 - 11t^2 + 6t)dt$$
$$= \left(5t^4 - \frac{11}{3}t^3 + 3t^2\right)\Big|_0^1$$
$$= \frac{13}{3}$$

11. Integrate the function $\vec{F} = x^2\hat{i} - xy\hat{j}$ from the point (0, 0) to (1, 1) along the parabola $y^2 = x$.

Solution.

Here the curve C is parabola $y^2 = x$ as shown in figure 4.13. On C, y can be taken as independent variable.

The dependent variable x can be written in terms of y as

$x = y^2$

$$dx = 2ydy$$

$$\vec{F}.d\vec{r} = x^2dx - xydy$$

$$= y^4.2ydy - y^2.\, ydy$$

$$= (2y^5 - y^3)dy$$

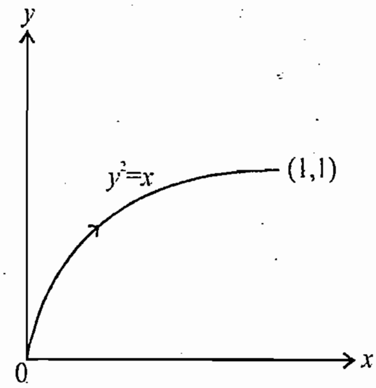


Figure 4.13

So, the line integral

$$\int_C \vec{F}.d\vec{r} = \int_0^1 (2y^5 - y^3)dy$$

$$= \frac{1}{3}y^6 - \frac{1}{4}y^4 \Big|_0^1 = \frac{1}{12}$$

12. Find the value of $\int_C [(x+y^2)dx + (x^2 - y)dy]$ taken in the counter-clockwise sense along the closed curve C formed by straight line $y = x$ and curve $y^3 = x^2$.

Solution.

The curve C consists of chord OA and curved part AO as shown in figure 4.14.

Equation of OA is $y = x$ and curved part is $y^3 = x^2$.

Along chord OA, x can be taken as independent variable and $y = x$.

$$\vec{F}.d\vec{r} = (x + y^2)dx + (x^2 - y)dy$$

$$= (x + x^2)dx + (x^2 - x)dx$$

$$= 2x^2dx$$

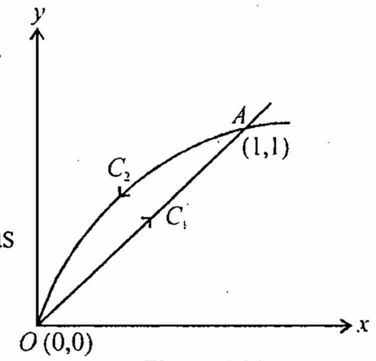


Figure 4.14

Along OA, x varies from 0 to 1. On curved part AO, let y be taken as independent variable & dependent variable x can be put as

$$x = y^{3/2},\ dx = \frac{3}{2}y^{1/2}dy.$$

$$\vec{F}.d\vec{r} = (x + y^2)dx + (x^2 - y)dy$$

$$= (y^{3/2} + y^2)\frac{3}{2}y^{1/2}dy + (y^3 - y)dy$$

$$= (y^3 + \frac{3}{2}y^{5/2} + \frac{3}{2}y^2 - y)dy$$

y varies from 1 to 0.

$$\oint_C \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r}$$

$$= \int_0^1 2x^2dx + \int_1^0 (y^3 + \frac{3}{2}y^{5/2} + \frac{3}{2}y^2 - y)dy$$

$$= \frac{2}{3}x^3\Big|_0^1 + \frac{1}{4}y^4 + \frac{3}{7}y^{7/2} + \frac{1}{2}y^3 - \frac{1}{2}y^2\Big|_1^0$$

$$= -\frac{1}{84}$$

Note:– If the integral is carried out in clockwise direction. The answer will differ only in sign.

$\oint_C \vec{F}.d\vec{r}$ in clockwise direction $= \frac{1}{84}$.

13. Calculate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = \frac{y^2}{x^2+y^2}\hat{i} - \frac{x^2}{x^2+y^2}\hat{j}$, where C is the semi-circle $y = \sqrt{a^2 - x^2}$.

Solution.

The curve C shown in Figure 4.15 is the semi-circle

$$y = \sqrt{a^2 - x^2}.$$

The equation can be written in parametric form as

$$x = a\cos\theta \Rightarrow dx = -a\sin\theta\, d\theta$$

$$y = a\sin\theta \Rightarrow dy = a\cos\theta\, d\theta$$

θ varies from 0 to π.

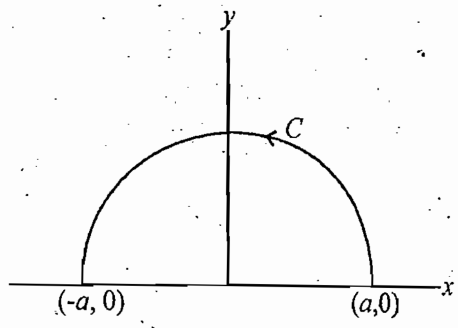


Figure 4.15

$$\vec{F}.d\vec{r} = \frac{y^2dx - x^2dy}{x^2+y^2}$$

$$= \frac{a^2\sin^2\theta(-a\sin\theta)d\theta - (a^2\cos^2\theta).a\cos\theta d\theta}{a^2}$$

$$= -a(\sin^3\theta + \cos^3\theta)d\theta$$

$$\int_C \vec{F}.d\vec{r} = -a\int_0^{\pi}(\sin^3\theta + \cos^3\theta)d\theta$$

$$= -a\int_0^{\pi}\sin^3\theta d\theta - a\int_0^{\pi}\cos^3\theta d\theta$$

$$= -2a\int_0^{\pi/2}\sin^3\theta d\theta - 0 \quad \left(\text{Since, } \int_0^{\pi}\cos^3\theta d\theta = 0\right)$$

$$= -2a\frac{\Gamma 2\,\Gamma 1/2}{2.\Gamma 5/2}$$

$$= -\frac{4a}{3}$$

14. Evaluate $\int_C \frac{dx}{x+y}$ where C is the curve $y^2 = 4ax$ from $(0, 0)$ to $(4a, 4a)$.

Solution.

The equation of curve (as shown in the figure 4.16) can be written in parametric form as

$$x = at^2 \Rightarrow dx = 2atdt$$

$$y = 2at \Rightarrow dy = 2adt$$

Parameter, t varies from 0 to 2.

The integral

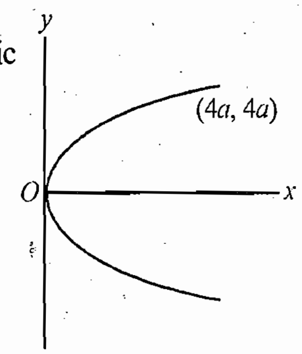


Figure 4.16

$$\int_0 \frac{dx}{x+y} = \int_0^2 \frac{2atdt}{at^2+2at} = 2\int_0^2\frac{dt}{t+2} = 2\log(t+2)\Big|_0^2$$

$$= 2\log 2$$

15. Evaluate $\int_C (ydx - xdy)$, where C is arc of cycloid $x = 2(\theta - \sin\theta)$, $y = 2(1-\cos\theta)$ joining the points $(0, 0)$ and $(4\pi, 0)$.

Solution.

The parametric equation of curve C as shown in figure 4.17 is given as

$$x = 2(\theta - \sin\theta) \Rightarrow dx = 2(1 - \cos\theta)d\theta$$
$$y = 2(1 - \cos\theta) \Rightarrow dy = 2\sin\theta\, d\theta$$

θ is the parameter since (x, y) varies from $(0, 0)$ to $(4\pi, 0)$. So, θ will vary from 0 to 2π.

The integrand $ydx - xdy$

$$= 2(1-\cos\theta).\,2(1-\cos\theta)\,d\theta - 2(\theta - \sin\theta).\,2\sin\theta\, d\theta$$
$$= 4\,(2 - 2\cos\theta - \theta\sin\theta)d\theta$$

So, $\displaystyle\int_C ydx - xdy = 4\int_0^{2\pi}(2 - 2\cos\theta - \theta\sin\theta)d\theta$

$$= 8\int_0^{2\pi} d\theta - 8\int_0^{2\pi}\cos\theta d\theta - 4\int_0^{2\pi}\theta\sin\theta d\theta$$
$$= 16\pi - 8[\sin\theta]_0^{2\pi} - 4[-\theta\cos\theta + \sin\theta]_0^{2\pi}$$
$$= 24\pi$$

16. Evaluate $\displaystyle\int_C \vec{F}.d\vec{r}$ where $\vec{F} = (2a - y)\hat{i} - (a - y)\hat{j}$ where C is the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from $(0, 0)$ to $(2\pi a, 0)$.

Solution.

The equation of cycloid is written in parametric form as

$$x = a(\theta - \sin\theta) \Rightarrow dx = a(1 - \cos\theta)d\theta$$
$$y = a(1 - \cos\theta) \Rightarrow dy = a\sin\theta\, d\theta$$

where θ is the parameter varying from 0 to 2π.

On C: $\vec{F}.d\vec{r} = (2a - y)dx - (a - y)dy$

$$= a\,(1 + \cos\theta).\,a\,(1 - \cos\theta)d\theta - a\cos\theta.a\sin\theta\, d\theta$$
$$= a^2\,(1 - \cos^2\theta - \sin\theta\cos\theta)d\theta$$

So, the line integral

$$\int_C \vec{F}.d\vec{r} = a^2\int_0^{2\pi}(1 - \cos^2\theta - \sin\theta\cos\theta)d\theta$$
$$= a^2\int_0^{2\pi} d\theta - a^2\int_0^{2\pi}\cos^2\theta d\theta - a^2\int_0^{2\pi}\sin\theta\cos\theta d\theta$$
$$= \pi a^2$$

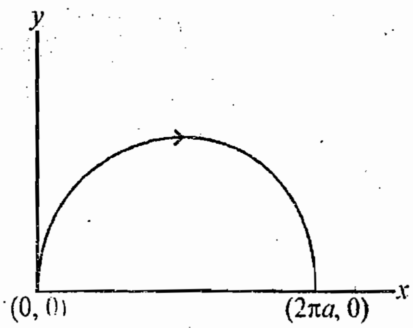


Figure 4.17

17. Evaluate $\displaystyle\int_C \frac{x^2dy - y^2dx}{x^{5/3} + y^{5/3}}$

where C is the quarter of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ from the point $(a, 0)$ to the point $(0, a)$.

Solution.

The parametric equation of the astroid as shown in figure 4.18 is given as

$$x = a\cos^3 t \Rightarrow dx = -3a\cos^2 t\sin t dt$$
$$y = a\sin^3 t \Rightarrow dy = 3a\sin^2 t\cos t dt$$

(x, y) varies from $(a, 0)$ to $(0, a)$.

So, t varies from 0 to $\pi/2$.

The integrand $\dfrac{x^2dy - y^2dx}{x^{5/3}+y^{5/3}}$

$$= \frac{a^2\cos^6 t(3a\sin^2 t\cos t)dt - (a^2\sin^6 t).(-3a\cos^2 t\sin t)dt}{a^{5/3}(\cos^5 t+\sin^5 t)}$$

$= 3a^{4/3}\sin^2 t\cos^2 t dt$

The line integral reduces to

$$3a^{4/3}\int_0^{\pi/2}\sin^2 t\cos^2 t\, dt = 3a^{4/3}\frac{\lceil 3/2\lceil 3/2}{2\lceil 3} = \frac{3\pi a^{4/3}}{16}$$

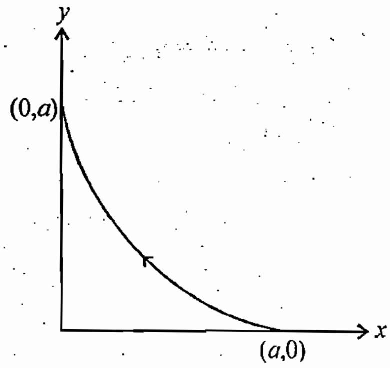


Figure 4.18

18. Find the value of $\int_c (x^2+y^2)dy$ taken in the counter clockwise sense along the quadrilateral with vertices (0, 0), (2, 0), (4, 4), (0, 4).

Solution.

The quadirlateral as shown in Figure 4.19 consists of four pieces of smooth curves *AB*, *BC*, *CD* & *DA*.

On *AB*, $y = 0$, $dy = 0$, $(x^2 + y^2)dy = 0$

On *BC*, $y - 0 = \dfrac{4-0}{4-2}(x-2)$

$y = 2x - 4$, $dy = 2dx$

$(x^2 + y^2)dy = (x^2 + (2x-4)^2)\, 2dx$

$= (10x^2 - 32x + 32)dx$

x varies from 2 to 4

On *CD*, $y = 4$, $dy = 0$

$(x^2 + y^2)dy = 0$

On *DA*, $x = 0$, $dx = 0$

$(x^2 + y^2)dy = y^2dy$

y varies from 4 to 0

So, the line integral

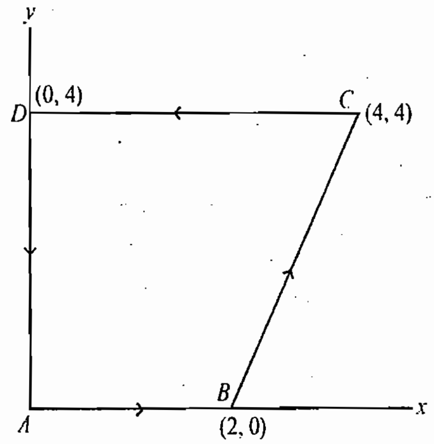


Figure 4.19

$$\int_C (x^2+y^2)dy = \int_{AB}(x^2+y^2)dy + \int_{BC}(x^2+y^2)dy + \int_{CD}(x^2+y^2)dy + \int_{DA}(x^2+y^2)dy$$

$$= 0 + \int_2^4 (10x^2 - 32x + 32)dx + 0 + \int_4^0 y^2dy$$

$$= \left(\frac{10}{3}x^3 - 16x^2 + 32x\right)\Bigg|_2^4 + \frac{y^3}{3}\Bigg|_4^0$$

$$= \frac{112}{3}$$

19. Evaluate $\int_C xy^2dy - x^2ydx$ taken in the counter clockwise sense along the cardioid $r = a(1+\cos\theta)$

Solution.

The curve C as shown in figure 4.20 cardiod whose equation is

$$r = a(1+\cos\theta)$$

$$x = r\cos\theta = a(1+\cos\theta)\cos\theta = a(\cos\theta + \cos^2\theta)$$

$$dx = a(-\sin\theta - 2\cos\theta\sin\theta)d\theta$$

$y = r\sin\theta = a(1+\cos\theta)\sin\theta = a(\sin\theta + \sin\theta\cos\theta)$

$dy = a(\cos\theta + \cos^2\theta - \sin^2\theta)d\theta$

The integrand

$(xy^2dy - x^2ydx)$

$= r^3\cos\theta\sin^2\theta\, a(\cos\theta + \cos^2\theta - \sin^2\theta)d\theta$

$\quad - r^3\cos^2\theta\sin\theta\, a(\sin\theta - 2\cos\theta\sin\theta)d\theta$

$= ar^3\cos\theta\sin^2\theta(\cos\theta + \cos^2\theta - \sin^2\theta + \cos\theta + 2\cos^2\theta)d\theta$

$= a^4(1+\cos\theta)^3\cos\theta\sin^2\theta(4\cos^2\theta + 2\cos\theta - 1)d\theta$

$= a^4[\cos^6\theta\sin^2\theta + 14\cos^5\theta\sin^2\theta + 17\cos^4\theta\sin^2\theta$

$\quad + 7\cos^3\theta\sin^2\theta - \cos^2\theta\sin^2\theta - \cos\theta\sin^2\theta]$

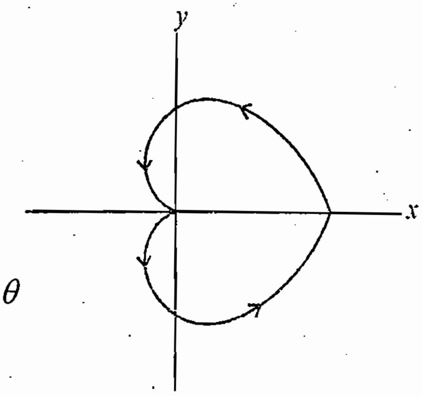


Figure 4.20

The line integral

$\oint(xy^2dy - x^2ydx)$

$$= a^4\int_0^{2\pi}(\cos^6\theta\sin^2\theta + 14\cos^5\theta\sin^2\theta + 17\cos^4\theta\sin^2\theta + 7\cos^3\theta\sin^2\theta - \cos^2\theta\sin^2\theta - \cos\theta\sin^2\theta)d\theta$$

$$= \frac{35}{16}a^4\pi$$

20. A particle moves counterclockwise along the curve $3x^2 + y^2 = 3$ from (1, 0) to a point P, under the action of the force

$$\vec{F}(x,y) = \frac{x}{y}\hat{i} + \frac{y}{x}\hat{j}.$$

Prove that there are two possible locations of P such that the work done by $\vec{F}$ is 1.

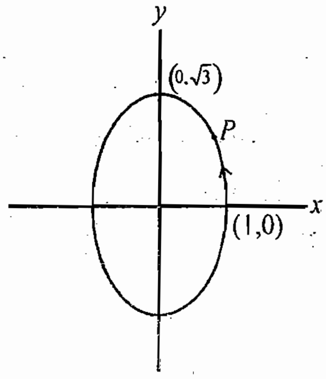


Figure 4.21

Solution. $\frac{x^2}{1} + \frac{y^2}{3} = 1$

Point on ellipse is represented as

$(\cos\theta, \sqrt{3}\sin\theta)$

$$\int \vec{F}.d\vec{r} = \int\left(\frac{x}{y}\hat{i} + \frac{y}{x}\hat{j}\right).(dx\hat{i} + dy\hat{j})$$

$$= \int\frac{x}{y}dx + \frac{y}{x}dy$$

$$= \int_0^\theta \frac{\cos\theta}{\sqrt{3}\sin\theta}(-\sin\theta)d\theta + \frac{\sqrt{3}\sin\theta}{\cos\theta}.\sqrt{3}\cos\theta d\theta$$

$$= \int_0^\theta\left(-\frac{1}{\sqrt{3}}\cos\theta + 3\sin\theta\right)d\theta$$

$$= -\frac{1}{\sqrt{3}}\sin\theta - 3\cos\theta\Big|_0^\theta$$

$$= -\frac{1}{\sqrt{3}}\sin\theta - 3\cos\theta + 3$$

Work done in equal to 1

So, $-\frac{1}{\sqrt{3}}\sin\theta-3\cos\theta+3=1$

$\Rightarrow \quad \frac{1}{\sqrt{3}}\sin\theta+3\cos\theta=2$

$\Rightarrow \quad \left(\frac{1}{\sqrt{3}}\sin\theta\right)^2=(2-3\cos\theta)^2$

$\Rightarrow \quad \frac{1}{3}\sin^2\theta=4+9\cos^2\theta-12\cos\theta$

$\Rightarrow \quad 28\cos^2\theta-36\cos\theta+11=0$

$\Rightarrow \quad (2\cos\theta-1)(14\cos\theta-11)=0$

$$\cos\theta=\frac{1}{2},\frac{11}{14}$$

So, there are two value of θ i.e., two possible location of P such that the work done by $\vec{F}$ is 1.

21. Find the circulation of the field

$$\vec{F}=-x^2y\hat{i}+xy^2\hat{j}+(y^3-x^3)\hat{k}$$

around the curve C, where C is the intersection of the sphere $x^2+y^2+z^2=25$ and the plane $z=3$. The orientation of the curve C is counterclockwise when viewed from above.

Solution.

$$\vec{F}=-x^2y\hat{i}+xy^2\hat{j}+(y^3-x^3)\hat{k}$$

C is the curve of intersection of surfaces

$x^2+y^2+z^2=25,\ z=3$

So, $\quad x^2+y^2=16$

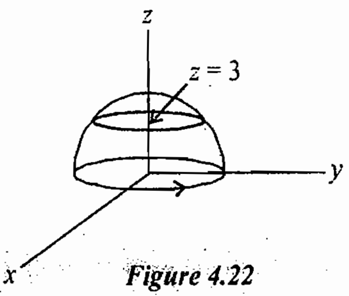


Figure 4.22

$\vec{F}\cdot d\vec{r}=x^2ydx+xy^2dy+(y^3-x^3)dz$

For curve C, $z=3$, $dz=0$

$$\oint\vec{F}\cdot d\vec{r}=\int -x^2ydx+xy^2dy$$

Let $x=4\cos\theta,\ y=4\sin\theta$

$$\oint\vec{F}\cdot d\vec{r}=\int_0^{2\pi}(256\cos^2\theta\sin^2\theta d\theta+256\cos^2\theta\sin^2\theta)d\theta$$

$$=512\int_0^{2\pi}\sin^2\theta\cos^2\theta d\theta$$

$$=512\times 4\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta$$

$$=2048\frac{\lceil 3/2\,\lceil 3/2}{2\lceil 3}=128\pi$$

22. If $\phi=2x^2yz$, $\vec{F}=xy\hat{i}-z^2y\hat{j}+x^2\hat{k}$ and C is the curve $x=2t$, $y=t^2$, $z=t^3$ from $t=0$ and $t=1$. Evaluate the line integrals (a) $\int_C\phi\, d\vec{r}$ (b) $\int_C\vec{F}\times d\vec{r}$.

Solution.

(a) Along C, $\phi = 2x^2yz = 2(2t)^2.t^2.t^3 = 8t^7$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= 2t\hat{i} + t^2\hat{j} + t^3\hat{k}$$
$$d\vec{r} = (2\hat{i} + 2t\hat{j} + 3t^2\hat{k})dt$$
$$\int_C \phi d\vec{r} = \int_0^1 8t^7(2\hat{i} + 2t\hat{j} + 3t^2\hat{k})dt$$
$$= \hat{i}\int_0^1 16t^7 dt + \hat{j}\int_0^1 16t^8 dt + \hat{k}\int_0^1 24t^9 dt$$
$$= 2\hat{i} + \frac{16}{9}\hat{j} + \frac{12}{5}\hat{k}$$

(b) Along C, $\vec{F} = xy\hat{i} - z^2y\hat{j} + x^2\hat{k}$

$$= 2t^3\hat{i} - t^8\hat{j} + 4t^2\hat{k}$$
$$\vec{F}\times d\vec{r} = (2t^3\hat{i} - t^8\hat{j} + 4t^2\hat{k})\times(2\hat{i} + 2t\hat{j} + 3t^2\hat{k})$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^8 & 4t^2 \\ 2 & 2t & 3t^2 \end{vmatrix}$$
$$= (-3t^{10} - 8t^3)\hat{i} + (8t^2 - 6t^3)\hat{j} + (4t^4 + 2t^8)\hat{k}$$
$$\int_C \vec{F}\times d\vec{r} = \hat{i}\int_0^1 (-3t^{10} - 8t^3)dt + \hat{j}\int_0^1 (8t^2 - 6t^3)dt + \hat{k}\int_0^1 (4t^4 + 2t^8)dt$$
$$= -\frac{47}{11}\hat{i} + \frac{5}{3}\hat{j} + \frac{46}{45}\hat{k}$$

23. Find the work done in moving the particle once round the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1, z = 0$ under the field of force given by $\vec{F} = (2x + y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$.

Solution

Work done moving the particle by distance dr

$$\vec{F}\cdot d\vec{r} = (2x + y + z)dx + (x + y - z^2)dy + (3x - 2y + 4z)dz$$

The curve C is ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

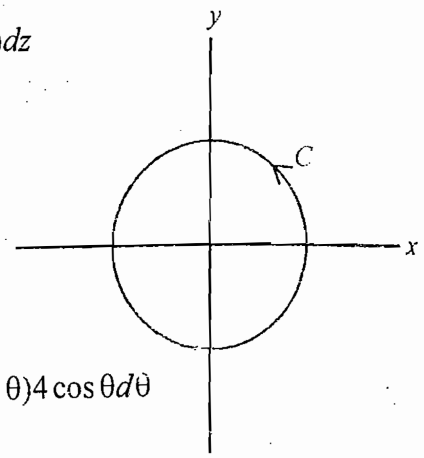


Figure 4.23

The equation of ellipse is given by $x = 5\cos\theta, y = 4\sin\theta, z = 0$

$$dx = -5\sin\theta d\theta$$
$$dy = 4\cos\theta d\theta$$
$$dz = 0$$
$$\vec{F}\cdot d\vec{r} = (10\cos\theta + 4\sin\theta)(-5\sin\theta)d\theta + (5\cos\theta + 4\sin\theta)4\cos\theta d\theta$$
$$= (-34\sin\theta\cos\theta + 20\cos^2\theta - 20\sin^2\theta)d\theta$$

On C, θ varies from 0 to 2π

So, work done in moving a particle around the ellipse

So, $W = \oint \vec{F} \cdot d\vec{r}$

$$= \int_0^{2\pi} (-34\sin\theta\cos\theta + 20\cos^2\theta - 20\sin^2\theta)d\theta$$

$$= -34\int_0^{2\pi} \sin\theta\cos\theta d\theta + 20\int_0^{2\pi} \cos^2\theta d\theta - 20\int_0^{2\pi} \sin^2\theta d\theta$$

$$= 0$$

24. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = c\left[-3a\sin^2\theta\cos\theta\hat{i} + a(2\sin\theta - 3\sin^2\theta)\hat{j} + b\sin 2\theta\hat{k}\right]$ and the curve C is given by $\vec{r} = a\cos\theta\hat{i} + a\sin\theta\hat{j} + b\theta\hat{k}$, θ varying from $\frac{\pi}{4}$ to $\frac{\pi}{2}$.

Solution.

$$\vec{r} = a\cos\theta\hat{i} + a\sin\theta\hat{j} + b\theta\hat{k}$$

$$d\vec{r} = (-a\sin\theta\hat{i} + a\cos\theta\hat{j} + b\hat{k})d\theta$$

$$\vec{F} \cdot d\vec{r} = c\left[3a^2\sin^3\theta\cos\theta + a^2(2\sin\theta - 3\sin^2\theta)\cos\theta + b^2\sin 2\theta\right]d\theta$$

The line integral

$$\int_C \vec{F} \cdot d\vec{r} = 3a^2c\int_{\pi/4}^{\pi/2} \sin^3\theta\cos\theta d\theta + a^2c\int_{\pi/4}^{\pi/2} (2\sin\theta - 3\sin^2\theta)\cos\theta d\theta + b^2c\int_{\pi/4}^{\pi/2} \sin 2\theta d\theta$$

$$= 3a^2c\left[\frac{\sin^4\theta}{4}\right]_{\pi/4}^{\pi/2} + a^2c\left[\sin^2\theta - \sin^3\theta\right]_{\pi/4}^{\pi/2} - \frac{b^2c}{2}\left[\cos 2\theta\right]_{\pi/4}^{\pi/2}$$

$$= \frac{9}{16}a^2c + a^2c\left[\frac{1}{2} - \frac{1}{2\sqrt{2}}\right] + \frac{b^2c}{2}$$

$$= \left(\frac{17}{16} - \frac{1}{2\sqrt{2}}\right)a^2c + \frac{b^2c}{2}$$

25. Evaluate

$\int_C (y^2 + z^2)dx + (z^2 + x^2)dy + (x^2 + y^2)dz$ where C is the part for which $z \geq 0$ of the intersection of the surfaces $x^2 + y^2 + z^2 = 4x$, $x^2 + y^2 = 2x$ and curve begins at the origin and runs at first in the positive octant.

Solution.

The C is the intersection of the two surfaces

$(x-1)^2 + y^2 = 1$ (Cylinder)

$z^2 = 2x$ (Parabolic cylinder)

The parametric equation of C is given as

$$x = 1 + \cos\theta = 2\cos^2\theta/2$$

$$dx = -2\sin\theta/_2 \cos\theta/_2 \, d\theta$$

$$y = \sin\theta = 2\sin\theta/_2 \cos\theta/_2$$

$$dy = \cos\theta d\theta$$

$$z = \sqrt{2(1+\cos\theta)} = 2\cos\theta/_2$$

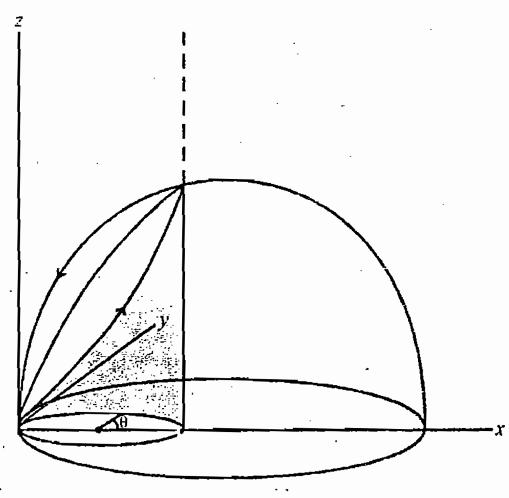


Figure 4.24

$$dz = -\sin \theta/_2\, d\theta$$

$(y^2 + z^2)\, dx + (z^2 + x^2)\, dy + (x^2 + y^2)\, dz$

$$= \left(4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} + 4\cos^2\frac{\theta}{2}\right)\left(-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta\right) + \left(4\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}\right)\cos\theta d\theta$$
$$+\left(4\cos^4\frac{\theta}{2} + 4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)\left(-\sin\frac{\theta}{2}\right)d\theta$$

So, the line integral becomes

$$\int_{-\pi}^{\pi} (y^2+z^2)dx + (z^2+x^2)\,dy + (x^2+y^2)dz$$

$$= -\int_{-\pi}^{\pi} 4\left(\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)\sin\theta d\theta + 4\int_{-\pi}^{\pi}\cos^2\frac{\theta}{2}\left(1+\cos^2\frac{\theta}{2}\right)\cos\theta d\theta$$
$$-4\int_{-\pi}^{\pi}\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}\,d\theta$$

The first and third integral vanishes since, the integrand is an odd function

So, integral reduces to

$$I = 4\int_{-\pi}^{\pi}\cos^2\frac{\theta}{2}\left(1+\cos^2\frac{\theta}{2}\right)\cos\theta d\theta$$

$$= \int_{-\pi}^{\pi}\left(2\cos^2\frac{\theta}{2}\right)\left(2+2\cos^2\frac{\theta}{2}\right)\cos\theta d\theta$$

$$= \int_{-\pi}^{\pi}(1+\cos\theta)\cdot(3+\cos\theta)\cos\theta d\theta$$

$$= \int_{-\pi}^{\pi}\cos^3\theta + 4\cos^2\theta + 3\cos\theta d\theta$$

$$= \int_{-\pi}^{\pi}\cos^3\theta d\theta + 4\int_{-\pi}^{\pi}\cos^2\theta d\theta + 3\int_{-\pi}^{\pi}\cos\theta d\theta$$

$$= 2\int_{0}^{\pi}\cos^3\theta d\theta + 16\int_{0}^{\pi/2}\cos^2\theta d\theta + 6\int_{0}^{\pi/2}\cos\theta d\theta$$

$$= 0 + 16\cdot\frac{\pi}{4} + 0 = 4\pi$$

26. Evaluate the following integrals along segment of straight line joining the given points

(i) $\int xdx + ydy + (x+y-1)dz$ from (1, 1, 1) to (2, 3, 4)

(ii) $\int \dfrac{xdx + ydy + zdz}{\sqrt{x^2+y^2+z^2-x-y+2z}}$ from (1, 1, 1) to (4, 4, 4)

Solution.

(i) The curve C is a line joining (1, 1, 1) to (2, 3, 4)

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3} = t \text{ (parameter)}$$

The parametric form of line is given as

$$x = 1 + t,\ y = 2t + 1,\ z = 3t + 1$$

$$dx = dt,\ dy = 2dt,\ dz = 3dt$$

t varies from 0 to 1

The line integral

$$I = \int xdx + ydy + (x + y - 1)\,dz$$

$$= \int d\left(\frac{x^2 + y^2}{2}\right) + \int (x + y - 1)\,dz$$

$$= \frac{x^2 + y^2}{2}\Bigg|_{(1,1,1)}^{(2,3,4)} + \int_0^1 (3t + 1)3dt$$

$$= \frac{1}{2} + 3\left[\frac{3}{2}t^2 + t\right]_0^1$$

$$= \frac{1}{2} + \frac{15}{2} = 8$$

(ii) The curve is straight line from (1, 1, 1) to (4, 4, 4) given by

$$x = t + 1,\ dx = dt$$

$$y = t + 1,\ dy = dt$$

$$z = t + 1,\ dz = dt$$

t varies from 0 to 3

The integral reduces to

$$\int_C \frac{xdx + ydy + zdz}{\sqrt{x^2 + y^2 + z^2 - x - y + 2z}} = \int_0^3 \frac{(t+1)dt + (t+1)dt + (t+1)dt}{\sqrt{3(t+1)^2}}$$

$$= \sqrt{3}\int_0^3 dt$$

$$= 3\sqrt{3}$$

27. Find the integral

$$\int_C (y + z)dx + (z + x)dy + (x + y)dz$$

where C is the circle $x^2 + y^2 + z^2 = a^2,\ x + y + z = 0$

Solution.

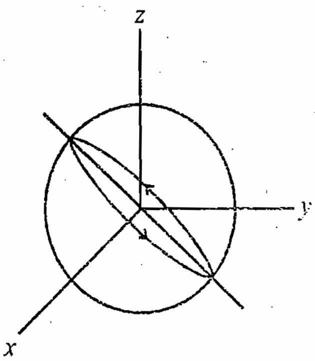


Figure 4.25

$$\oint (y + z)dx + (z + x)dy + (x + y)dz$$

$$= \oint ydx + zdx + zdy + xdy + xdz + ydz$$

$$= \oint d(xy + yz + zx) = 0$$

The integral is an exact differential

So, $\oint \vec{F} \cdot d\vec{r} = 0$

28. Evaluate

$$\int_C x^2 y^3 dx + dy + zdz$$

where C is the circle $x^2 + y^2 = R^2,\ z = 0$

Solution.

The curve C is the circle $x^2 + y^2 = R^2, z = 0$

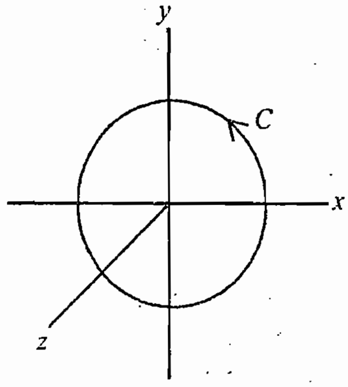


Figure 4.26

$$x = R\cos\theta,\ dx = R\sin\theta d\theta$$
$$y = R\sin\theta,\ dy = R\cos\theta d\theta$$
$$I = \oint (x^2y^3dx + dy + zdz)$$
$$= \oint x^2y^3dx + \oint (dy + zdz)$$
$$= \int_0^{2\pi} R^2\cos^2\theta \cdot R^3\sin^3\theta(-R\sin\theta)d\theta + \oint d\left(y + \frac{z^2}{2}\right)$$
$$= -R^6\int_0^{2\pi} \cos^2\theta\sin^4\theta d\theta + 0$$
$$= -4R^6\int_0^{\pi/2} \sin^4\theta\cos^2\theta d\theta$$
$$= -4R^6\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma 4}$$
$$= -\frac{\pi R^6}{8}$$

29. Evaluate $\int_C \vec{A}\cdot d\vec{r}$ along the curve $x^2 + y^2 = 1, z = 1$ from (0, 1, 1) to (1, 0, 1) if

$$\vec{A} = (yz + 2x)\hat{i} + xz\hat{j} + (xy + 2z)\hat{k}$$

Solution.

The curve C is the circle of radius 1 with the centre at (0, 0, 1) lying in a plane parallel to xy plane.

$$\vec{F}\cdot d\vec{r} = (yz + 2x)dx + xzdy + (xy + 2z)dz$$
$$= d(xyz + x^2 + z^2)$$

$\vec{F}\cdot d\vec{r}$ is an exact differential. So, line integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of curve joining initial and final points

$$\int_C \vec{F}\cdot d\vec{r} = \int d(xyz + x^2 + z^2)$$
$$= \left[xyz + x^2 + z^2\right]_{(0,1,1)}^{(1,0,1)} = 1$$

30. Evaluate $\int_c yzdx + zxdy + xydz$ where C is the are of curve $x = bcost, y = bsint, z = \frac{at}{2\pi}$ from the point it intersects $z = 0$ to the point it intersects $z = a$.

Solution.

The curve C is a spiral given by

$$x = bcost,\ y = bsint,\ z = \frac{at}{2\pi}$$

Since, z varies from $z = 0$ to $z = a$, hence, t varies from 0 to 2π

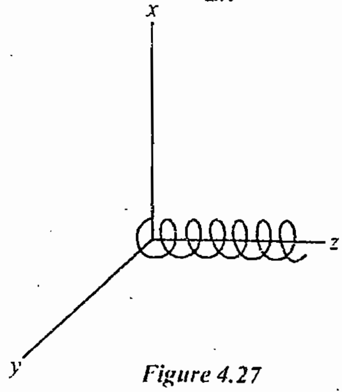


Figure 4.27

The line integral

$$\int_c (yzdx + zxdy + xydz)$$

$$= \int_c d(xyz) = [xyz]$$

$$= \left[\frac{ab^2}{2\pi} t \sin t \cos t\right]_0^{2\pi} = 0$$

31. Evaluate $\int_C y^2dx + z^2dy + x^2dz$ where C is the curve of intersention of the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$ $(a>0, z\geq 0)$ integrated anticlockwise when viewed from the origin.

Solution.

The curve C is the curve of intersection of $x^2+y^2=ax \Rightarrow \left(x-\frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}, x^2+y^2+z^2=a^2$

$$x^2+y^2+z^2=a^2$$

$$\Rightarrow \quad z^2 + ax = a^2$$

$$\Rightarrow \quad z^2 = -a(x-a)$$

Let $x = \frac{a}{2}+\frac{a}{2}\cos\theta \quad \Rightarrow dx = -\frac{a}{2}\sin\theta d\theta$

$y = \frac{a}{2}\sin\theta \quad \Rightarrow dy = \frac{a}{2}\cos\theta d\theta$

$$z^2 = a(a-x)$$

$$= a\left(\frac{a}{2}-\frac{a}{2}\cos\theta\right)$$

$$= a^2\sin^2\frac{\theta}{2}$$

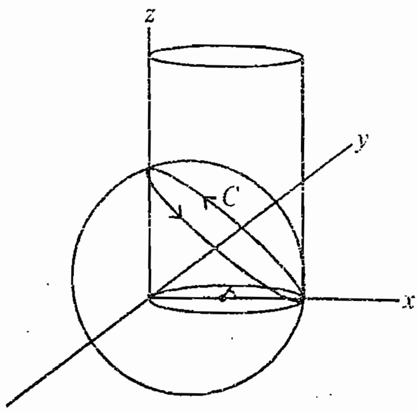


Figure 4.28

So, $z = a\sin\frac{\theta}{2} \quad \Rightarrow dz = \frac{a}{2}\cos\frac{\theta}{2}d\theta$

θ varies from 0 to 2π.

The line integral

$$I = \int y^2dx + z^2dy + x^2dz$$

$$= \int_0^{2\pi} \frac{a^2}{4}\sin^2\theta\left(-\frac{a}{2}\sin\theta d\theta\right)d\theta + \int_0^{2\pi}\frac{a^2}{2}(1-\cos\theta)\frac{a}{2}\cos\theta d\theta + \int_0^{2\pi}\frac{a^2}{4}(1+\cos\theta)^2\frac{a}{2}\cos\theta/2\, d\theta$$

$$= 0 + \frac{a^3}{4}\int_0^{2\pi}(\cos\theta - \cos^2\theta)d\theta + \frac{a^3}{2}\int_0^{2\pi}\cos^5\theta/2\, d\theta$$

$$= \frac{a^3}{2}\int_0^{\pi}\cos\theta d\theta - \frac{a^3}{2}\int_0^{\pi}\cos^2\theta d\theta + \frac{a^3}{2}\int_0^{2\pi}\cos^5\frac{\theta}{2}d\theta$$

$$= -a^3\int_0^{\pi/2}\cos^2\theta d\theta + a^3\int_0^{\pi}\cos^5\phi d\phi \qquad (\phi = 0/2)$$

$$= -\frac{a^3\pi}{4} + 0$$

$$= -\frac{a^3\pi}{4}$$

EXERCISE – 1

1. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F}$ is $x^2y^2\hat{i} + y\hat{j}$ and C is $y^2 = 4x$ in the xy–plane from (0, 0) to (4, 4).

2. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = [-3a\sin^2 t\cos t\hat{i} + a(2\sin t - 3\sin^3 t)\hat{j} + b\sin 2t\hat{k}]$ and C is given by $\vec{r} = a\cos t\hat{i} + a\sin t\hat{j} + bt\hat{k}$ from $t = \dfrac{\pi}{4}$ to $\dfrac{\pi}{2}$.

3. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ and C is the arc of curve $\vec{r} = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$ from $t = 0$ to $t = 2\pi$.

4. Find work done in moving a particle in a force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along a line joining (0, 0, 0) to (2, 1, 2).

5. If $\vec{A} = (2y + 3)\hat{i} + xz\hat{j} + (yz - x)\hat{k}$, evaluate $\int_C \vec{A}.d\vec{r}$ along the following paths C
   (a) $x = 2t^2, y = t, z = t^3$ from $t = 0$ to $t = 1$.
   (b) the straight lines from (0, 0, 0) to (0, 0, 1) then to (0, 1, 1) and then to (2, 1, 1).
   (c) the straight line joining (0, 0, 0) and (2, 1, 1)

6. Evaluate $\int_C \vec{A}.d\vec{r}$ where $\vec{A} = (y - z + z)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}$ over a closed loop. C, in the form of a square of length 3 on the xy plane with its sides parallel to the x and y axis and one of the x vertices being $x = 0, y = 0, z = 0$.

7. If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - x)\hat{j}$. Evaluate $\int_C \vec{F}.d\vec{r}$ along the curve C in xy plane, $y = x^3$ from point (1, 1) to (2, 8).

8. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + 2\hat{k}$ along
   (a) the straight line from (0, 0, 0) to (2, 3, 5)
   (b) the space curve $x = t^2, y = 2t, z = t^2 - t$, from $t = 0$ to $t = 1$

9. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = (x - 2y)\hat{i} + (y - 2x)\hat{j}$ along the closed curve C in xy plane given by $x = \cos t, y = 2\sin t$ from $t = 0$ to $t = 2\pi$.

10. If $\vec{F} = (x + y^2)\hat{i} + (2y - x)\hat{j}$. Evaluate $\oint \vec{F}.d\vec{r}$ around a triangle C given in figure 4.29.

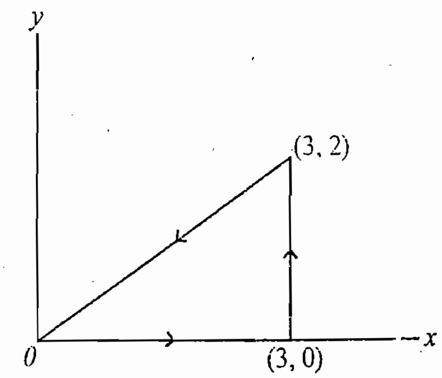


Figure 4.29

11. Evaluate $\oint \vec{F}.d\vec{r}$ around a closed curve C as shown in figure 4.30. if $\vec{F} = (x+y)\hat{i} + (x-y)\hat{j}$.

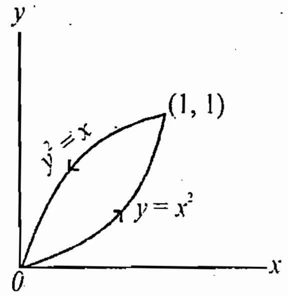


Figure 4.30

12. Find the circulation of $\vec{F}$ round the curve C where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and C is the circle of $x^2 + y^2 = a^2$, $z = 0$.

ANSWER

1. 264	**2.** $\frac{c}{2}(a^2+b^2)$	**3.** 3π	**4.** 35	**5.** **(a)** $\frac{288}{35}$, **(b)** 10, **(c)** 8
6. 9	**7.** $\frac{269}{4}$	**8.** **(a)** $\frac{67}{2}$, **(b)** $\left(-\frac{6}{5}\right)$	**9.** 0	**10.** –7
11. 0	**12.** πa^2			

Aryabhatta

Aryabhatta (476-550 A.D.) was born in Patliputra in Magadha, modern Patna in Bihar where he wrote his famous treatise the "Aryabhatta siddhanta" but more famously the "Aryabhatiya", the only work to have survived. It contains mathematical and astronomical theories that have been revealed to be quite accurate in modern mathematics. For instance he wrote that if 4 is added to 100 and then multiplied by 8 then added to 62,000 then divided by 20,000 the answer will be equal to the circumference of a circle of diameter twenty thousand. This calculates to 3.1416 close to the actual value Pi (3.14159). But his greatest contribution has to be zero. His other works include algebra, arithmetic, trigonometry, quadratic equations and the sine table.

He already knew that the earth spins on its axis, the earth moves round the sun and the moon rotates round the earth. He talks about the position of the planets in relation to its movement around the sun. He refers to the light of the planets and the moon as reflection from the sun. He goes as far as to explain the eclipse of the moon and the sun, day and night, the contours of the earth, the length of the year exactly as 365 days.

He even computed the circumference of the earth as 24835 miles which is close to modern day calculation of 24900 miles.

5

GREEN'S THEOREM

GREEN'S THEOREM

Let R be a closed bounded regions in the x–y plane whose boundary C consists of finitely many smooth curves. Let M and N be continuous function of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R. Then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint Mdx + Ndy$$

the line integral being taken along the entire boundary C of R such that R is on the left as one advances in the direction of integration.

Proof: We will prove the theorem in the case for a *simple* region. R, that is, where the boundary curve C can be written as $C = C_1 \cup C_2$ in two distinct ways:

C_1 = the curve $y = y_1(x)$ from the point X_1 to the point X_2 (1)

C_2 = the curve $y = y_2(x)$ from the point X_2 to the point X_1, (2)

where X_1 and X_2 are the points on C farthest to the left and right, respectively; and

C_1 = the curve $x = x_1(y)$ from the point Y_2 to the point Y_1 (3)

C_2 = the curve $x = x_2(y)$ from the point Y_1 to the point Y_2, (4)

where Y_1 and Y_2 are the lowest and highest points, respectively, on C. See Figure 5.1.

Integrate $M(x, y)$ around C using the representation $C = C_1 \cup C_2$ given by (1) and (2).

Since $y = y_1(x)$ along C_1, (as x goes from a to b) and $y = y_2(x)$ along C_2 (as x goes from b to a), as we see from Figure 5.1, then we have

$$\oint_C M(x,y)dx = \int_{C_1} M(x,y)dx + \int_{C_2} M(x,y)dx$$

$$= \int_a^b M(x,y_1(x))dx + \int_b^a M(x,y_2(x))dx$$

$$= \int_a^b M(x,y_1(x))dx - \int_a^b M(x,y_2(x))dx$$

$$= -\int_a^b M(x,y_2(x)) - M(x,y_1(x))dx$$

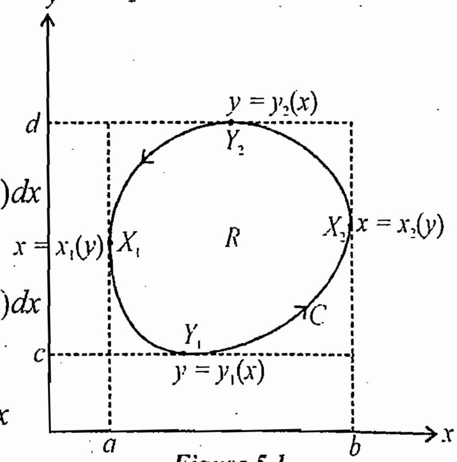


Figure 5.1

$$= -\int_a^b \left(M(x, y(x)\Big|_{y=y_1(x)}^{y=y_2(x)} \right) dx$$

$$= -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial M(x,y)}{\partial y} dydx \qquad \text{(by the Fundamental Theorem of Calculus)}$$

$$= -\iint_R \frac{\partial M}{\partial y} dA.$$

Likewise, integrate $N(x, y)$ around C using the representation $C = C_1 \cup C_2$ given by (3) and (4). Since $x = x_1(y)$ along C_1 (as y goes from d to c) and $x = x_2(y)$ along $C_2(y)$ (as y goes from c to d), as we see from Figure 5.1, then we have

$$\oint_C N(x,y)dy = \int_{C_1} N(x,y)dy + \int_{C_2} N(x,y)dy$$

$$= \int_d^c N(x_1(y), y)dy + \int_c^d N(x_2(y)y)dy$$

$$= -\int_c^d N(x_1(y)y)dy + \int_c^d N(x_2(y)y)dy$$

$$= \int_c^d (N(x_2(y),(y) - N(x_1(y), y))dy$$

$$= \int_c^d \left(N(x, y\Big|_{x=x_1(y)}^{x=x_2(y)} \right) dy$$

$$= \int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial N(x,y)}{\partial x} dxdy \text{ (by the Fundamental Theorem of Calculus)}$$

$$= \iint_R \frac{\partial N}{\partial x} dA,$$

So, $$\oint \vec{F}.d\vec{r} = \oint_C M(x,y)dx + \oint_C N(x,y)dy$$

$$= -\iint_R \frac{\partial M}{\partial y} dA + \iint_R \frac{\partial N}{\partial x} dA$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

There we proved Green's theorem only for a simple region R, the theorem can also be proved for more general regions (say, a union of simple regions).

1. Verify Green's theorem in the plane for $\oint_C (xy + x^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $x^2 = 4ay$.

Solution.

Here $Mdx + Ndy$

$= (xy+x^2)dx + x^2dy$

$$M = xy + x^2 \Rightarrow \frac{\partial M}{\partial y} = x$$

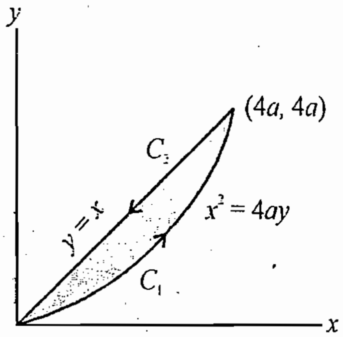


Figure 5.2

$$N = x^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$$

Let us first evaluate the double integral over Region R bounded by $x^2 = 4ay$ (curve C_1) & $y = x$ (curve C_2) as

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \int_0^{4a} \int_{x^2/4a}^{x} xdydx$$

$$= \int_0^{4a} x\left(x - \frac{x^2}{4a}\right) dx = \left.\frac{x^3}{3} - \frac{x^4}{16a}\right|_0^{4a} = \frac{16a^3}{3}$$

Now let us evaluate the line integral $\oint Mdx + Ndy$ on closed curve C. The curve C is a piecewise smooth curve consisting of C_1 and C_2.

On C_1, $y = \dfrac{x^2}{4a}$, $\quad dy = \dfrac{x}{2a} dx$

$$Mdx + Ndy = (xy + x^2)dx + x^2dy$$

$$= \left(\frac{x^3}{4a} + x^2 \right) dx + x^2 \frac{x}{2a} dx$$

$$= \left(\frac{3}{4} \cdot \frac{x^3}{a} + x^2 \right) dx$$

x varies from 0 to $4a$ on C_1.

So, $$\int_{C_1} Mdx + Ndy = \int_0^{4a} \left(\frac{3x^3}{4a} + x^2 \right) dx$$

$$= \left. \frac{3}{16a} x^4 + \frac{x^3}{3} \right|_0^{4a}$$

$$= 8a^3 + \frac{64a^3}{3} = \frac{208a^3}{3}$$

On C_2, $\quad y = x$, $dy = dx$

$$Mdx + Ndy = (xy + x^2)dx + x^2dy$$

$$= 3x^2dx$$

x varies from $4a$ to 0.

So, $$\int_{C_2} Mdx + Ndy = \int_{4a}^{0} 3x^2dx$$

$$= \left. x^3 \right|_{4a}^{0} = -64a^3$$

So, $$\int_{C} Mdx + Ndy = \int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy$$

$$= \frac{208}{3}a^3 - 64a^3 = \frac{16}{3}a^3$$

Since, $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint_C Mdx + Ndy$$

So, Green's theorem is verified.

2. Verify Green's theorem in the plane for $\oint_C (2xy-x^2)dx+(x^2+y^2)dy$ where C is the boundary of the region enclosed by $y=x^2$ and $y^2=x$ described in positive sense.

Solution

Here, $\quad Mdx + Ndy = (2xy - x^2)dx + (x^2 + y^2)\, dy$

$$M = 2xy - x^2 \Rightarrow \frac{\partial M}{\partial y} = 2x$$

$$N = x^2 + y^2 \quad \Rightarrow \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

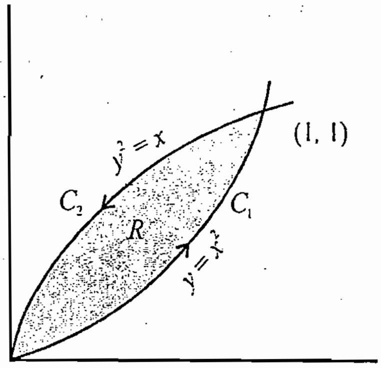


Figure 5.3

So, the double integral $\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$ over region R bounded by $y = x^2$ (curve C_1) and $y^2 = x$ (curve C_2) is zero as integer and $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

Now, let us evaluate the line integral over a closed curve C. The curve C is a piecewise smooth curve consisting of C_1 & C_2.

On $\quad C_1$, $y = x^2$, $dy = 2xdx$ (Taking x as independent variable).

$$\begin{aligned} Mdx + Ndy &= (2xy - x^2)dx + (x^2 + y^2)dy \\ &= (2x^3 - x^2)dx + (x^2 + x^4).2xdx \\ &= (2x^5 + 4x^3 - x^2)\, dx \end{aligned}$$

x varies from 0 to 1 on C_1

So, $$\int_{C_1} Mdx + Ndy = \int_0^1 (2x^5 + 4x^3 - x^2)dx$$

$$= \left.\frac{x^6}{3} + x^4 - \frac{x^3}{3}\right|_0^1 = 1$$

On C_2, $\quad x = y^2$, $dx = 2ydy$

Taking y as independent variable

$$\begin{aligned} Mdx + Ndy &= (2xy - x^2)dx + (x^2 + y^2)dy \\ &= (2y^2y - y^4).2ydy + (y^4 + y^2)dy \\ &= (-2y^5 + 5y^4 + y^2)dy \end{aligned}$$

y varies from 1 to 0 on C_2

$$\int_{C_2} Mdx + Ndy = \int_1^0 (-2y^5 + 5y^4 + y^2)dy$$

$$= -\left.\frac{y^6}{3} + y^5 - \frac{y^3}{3}\right|_1^0 = -1$$

So, $$\int_C Mdx + Ndy = \int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy = 0$$

Since, $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C Mdx + Ndy$$

So, Green's Theorem is verified.

3. Apply Green's theorem in the plane to evaluate $\oint_C \{(y-\sin x)dx+\cos x dy\}$ where C is the triangle enclosed by the lines $y=0, x=\pi, \pi y=2x$.

Solution

Here, $Mdx + Ndy = (y-\sin x)dx + \cos x dy$

So, $M = y-\sin x, \quad \dfrac{\partial M}{\partial y}=1$

$N=\cos x, \quad \dfrac{\partial N}{\partial x}=-\sin x$

According to Green's theorem

$$\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

where R is the region enclosed by the piecewise smooth curve C consisting of curve C_1 ($y=0$), curve C_2 ($x=\pi$) curve C_3 ($\pi y=2x$) as shown in Figure 5.4.

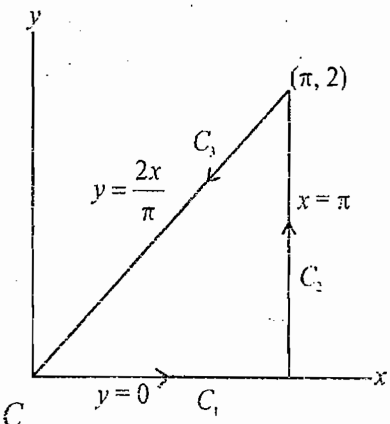


Figure 5.4

So, $$\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\int_0^2\int_{\pi y/2}^{\pi}(-\sin x-1)dxdy$$

$$=\int_0^2[\cos x-x]_{\pi y/2}^{\pi}dy$$

$$=\int_0^2\left(-1-\pi-\cos\frac{\pi y}{2}+\frac{\pi y}{2}\right)dy$$

$$=-(1+\pi)y-\frac{2}{\pi}\sin\frac{\pi y}{2}+\frac{\pi y^2}{4}\Bigg|_0^2=-2-\pi$$

4. If $\vec{F}=(x^2-y^2)\hat{i}+2xy\hat{j}$ and $\vec{r}=x\hat{i}+y\hat{j}$, find the value of $\oint(x^2-y^2)dx+2xydy$ around the rectangular boundary $x=0, x=a, y=0$ and $y=b$.

Solution

Here, the curve C is a piecewise smooth curve consisting of C_1 ($y=0$), C_2 ($x=a$), C_3 ($y=b$) & C_4 ($x=0$).

The region bounded by C is shown in figure 5.5.

$$\oint\vec{F}.d\vec{r}=\oint(x^2-y^2)dx+2xydy=\oint Mdx+Ndy$$

Here, $M=x^2-y^2, \quad \dfrac{\partial M}{\partial y}=-2y$

$N=2xy, \quad \dfrac{\partial N}{\partial x}=2y$

Applying Green's theorem

$$\oint Mdx+Ndy=\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

$$=4\int_0^b\int_0^a ydxdy=4a\int_0^b ydy$$

$$=2ab^2$$

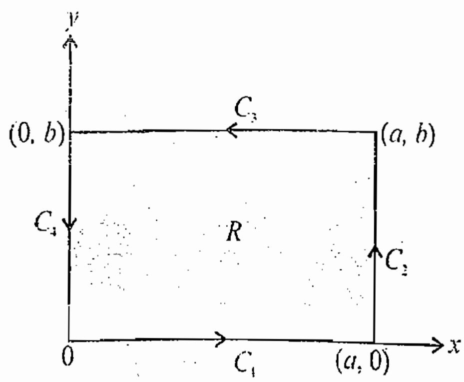


Figure 5.5

5. Evaluate $\oint_C e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$ by Green's theorem in plane where C is the rectangle with vertices $(0, 0), (\pi, 0), (\pi, \pi/2), (0, \pi/2)$.

Solution

The curve C is a piecewise smooth curve consisting of $C_1(y = 0)$, $C_2(x = \pi)$, $C_3(y = \pi/2)$ & $C_4(x = 0)$.

The region R bounded by C is as shown in figure.

$$\oint_C M dx + N dy = \oint_C e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$$

Here $M = e^{-x}\sin y \Rightarrow \dfrac{\partial M}{\partial y} = e^{-x}\cos y$

$N = e^{-x}\cos y \Rightarrow \dfrac{\partial N}{\partial x} = -e^{-x}\cos y$

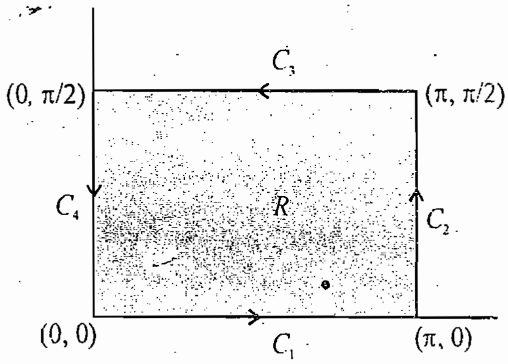


Figure 5.6

Applying Green's theorem

$$\oint M dx + N dy = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dydx$$

$$= -2\int_0^{\pi}\int_0^{\pi/2} e^{-x}\cos y\,dydx$$

$$= -2\int_0^{\pi} e^{-x}[\sin y]_0^{\pi/2}\,dx$$

$$= -2\int_0^{\pi} e^{-x}dx = 2e^{-x}\Big|_0^{\pi}$$

$$= 2(e^{-\pi} - 1)$$

6. Verify Green's theorem in the plane for $\oint (x^2 - x^3)dx + (y^2 - 2xy)dy$ where C is the square with vertices (0, 0), (2, 0), (2, 2) (0, 2).

Solution.

$M = x^2 - xy^3 \Rightarrow \dfrac{\partial M}{\partial y} = -3xy^2$

$N = y^2 - 2xy \Rightarrow \dfrac{\partial N}{\partial x} = -2y$

Let us first evaluate the double integral over region R bounded by curves $C_1(y = 0)$, $C_2(x = 2)$, $C_3(y = 2)$, $C_4(x = 0)$.

$$\iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dydx = \int_0^2\int_0^2 (3xy^2 - 2y)dydx$$

$$= \int_0^2 [xy^3 - y^2]_0^2 dx = 4\int_0^2 (2x - 1)dx$$

$$= 4[x^2 - x]_0^2 = 8$$

Now, let us evaluate the line integral as closed curve C. The curve C is a piecewise smooth curve consisting of C_1, C_2, C_3 & C_4.

On C_1: $y = 0$, $dy = 0$

$$M dx + N dy = x^2 dx$$

x varies from 0 to 2 on C_1

$$\int_{C_1} Mdx + Ndy = \int_0^2 x^2 dx = \frac{x^3}{3}\bigg|_0^2 = \frac{8}{3}$$

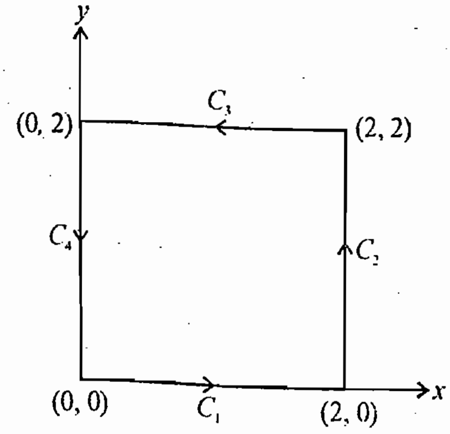


Figure 5.7

On C_2, $x = 2$, $dx = 0$

$$Mdx + Ndy = (y^2 - 4y)dy$$

y varies from 0 to 2 on C_2

$$\int_{C_2} Mdx + Ndy = \int_0^2 (y^2 - 4y)dy$$

$$= \frac{y^3}{3} - 2y^2\bigg|_0^2 = -\frac{16}{3}$$

On C_3, $y = 2$, $dy = 0$

$$Mdx + Ndy = (x^2 - 8x)dx$$

x varies from 2 to 0 on C_3

$$\int_{C_3} Mdx + Ndy = \int_2^0 (x^2 - 8x)dx$$

$$= \frac{x^3}{3} - 4x^2\bigg|_2^0 = \frac{40}{3}$$

On C_4, $x = 0$, $dx = 0$

$$Mdx + Ndy = y^2 dy$$

y varies from 2 to 0 on C_4

$$\int_{C_4} Mdx + Ndy = \int_2^0 y^2 dy$$

$$= \frac{y^3}{3}\bigg|_2^0 = -\frac{8}{3}$$

$$\int_C Mdx + Ndy = \int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy + \int_{C_3} Mdx + Ndy + \int_{C_4} Mdx + Ndy$$

$$= \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8$$

Since, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C Mdx + Ndy$

So, Green's theorem is verified.

7. Use Green's theorem to evaluate the integral $\oint_C x^2 dx + (x + y^2)dy$, where C is the closed curve given by $y = 0$, $y = x$ and $y^2 = 2 - x$ in the first quadrant, oriented counter clockwise.

Solution :

The given integral is

$$\oint x^2 dx + (x + y^2)dy = \oint Mdx + Ndy$$

So, $\quad M = x^2$; $N = x + y^2$

According to Green's theorem

$$\oint Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

(R is the region of integration as shown in Figure 5.8)

So, $\oint x^2dx + (x + y^2)dy = \iint dxdy$

$$= \int_0^1 \int_y^{2-y^2} dxdy$$

$$= \int_0^1 (2 - y^2 - y)dy$$

$$= \left[2y - \frac{y^3}{3} - \frac{y^2}{2}\right]_0^1 = \frac{7}{6}$$

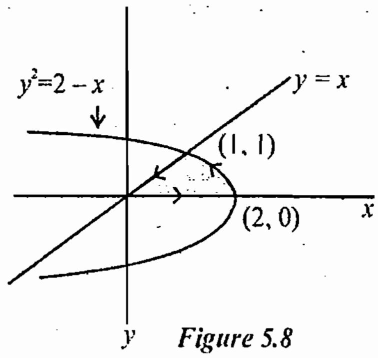


Figure 5.8

8. Let $\vec{F} = (x^2 - xy^2)\hat{i} + y^2\hat{j}$: Using Green's theorem, evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the positively oriented closed curve which is the boundary of the region enclosed by the x-axis and the semi-circle $y = \sqrt{1-x^2}$ in the upper half plane.

Solution.

$$\vec{F} = (x^2 - xy^2)\hat{i} + y^2\hat{j}$$

So, $\vec{F} \cdot d\vec{r} = (x^2 - xy^2)\hat{i} + y^2\hat{j}$

According to Green's theorem

$$\oint_C \vec{F}.d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

(R is the region of integration shown in Figure 5.9)

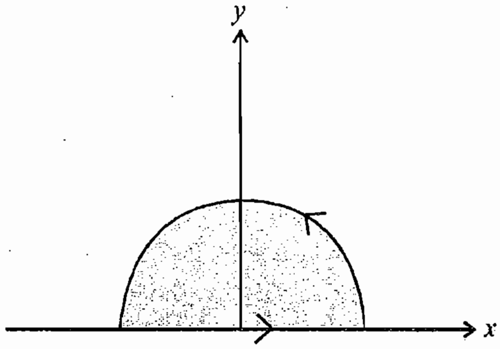


Figure 5.9

$$= \int_{-1}^{} \int_0^{\sqrt{1-x^2}} 2xydydx$$

$$= \int_{-1}^{1} x[y^2]_0^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} x(1 - x^2)\, dx = 0 \quad \left(\int_{-a}^{a} f(x) = 0 \text{ if } f(x) \text{ is odd function}\right)$$

9. Evaluate by Green's theorem $\oint (\cos x \sin y - xy)dx + \sin x \cos y dy$ where C is the circle $x^2 + y^2 = a^2$.

Solution.

The given integral is

$$\oint_C (\cos x \sin y - xy)dx + \sin x \cos y\, dy$$

Where curve C is a circle of radius a and centred at origin enclosing region R as shown in Figure 5.10.

Here $M = \cos x \sin y - xy \Rightarrow \frac{\partial M}{\partial y} = \cos x \cos y - x$

$N = \sin x \cos y \quad \Rightarrow \frac{\partial N}{\partial x} = \cos x \cos y$

Using Green's theorem

$$\oint Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint x\,dx\,dy$$

$$= \int_0^a \int_0^{2\pi} r\cos\theta\, r\,d\theta\, dr$$

$$= \int_0^a r^2[\sin\theta]_0^{2\pi}\, dr = 0$$

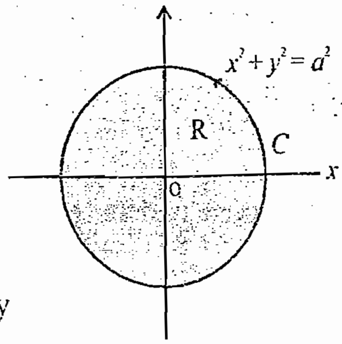


Figure 5.10

10. Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint x\,dy - y\,dx$. Hence find the area of the ellipse $x = a\cos\theta, y = b\sin\theta$

Solution

According to Green's theorem, if R is a plane region bounded by a simple closed curve C as shown in Figure 5.11 according to the Green's Theorem

$$\iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \oint M\,dx + N\,dy$$

Let us put $M = -y/2, N = x/2$

So, $$\frac{1}{2}\oint x\,dy - y\,dx = \iint \frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{2}\right)dx\,dy$$

$$= \iint dx\,dy$$

= Area of region R bounded by C.

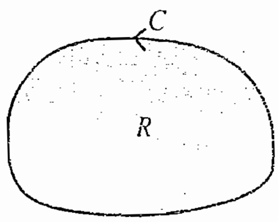


Figure 5.11

So, area of region bounded by simple closed curve C is given by

$$\frac{1}{2}\oint x\,dy - y\,dx$$

For as ellipse, $x = a\cos\theta \Rightarrow dx = -a\sin\theta\, d\theta$

$y = a\sin\theta \Rightarrow dy = a\cos\theta\, d\theta$

$x\,dy - y\,dx = a\cos\theta\, b\cos\theta - b\sin\theta\,(-a\sin\theta)d\theta = ab\,d\theta.$

So, area of region bounded by ellipse

$$= \frac{1}{2}\oint x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,d\theta$$

$$= \frac{1}{2}ab\int_0^{2\pi} d\theta = \pi ab$$

11. Apply Green's theorem in the plane to evaluate $\oint_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ where C is the boundary of the surface enclosed by the x–axis and the semi circle $y = \sqrt{a^2 - x^2}$.

Solution

Curve C is a piecewise smooth curve consisting of semi circle $C_1\left(y = \sqrt{a^2 - x^2}\right)$ & part of x axis C_2 $(y = 0)$.

Region R is bounded by curve C as shown in Figure 5.12.

$$\oint (2x^2 - y^2)dx + (x^2 + y^2)dy = \oint M\,dx + N\,dy$$

Here $M = 2x^2 - y^2, \frac{\partial M}{\partial y} = -2y$

$N = x^2 + y^2, \quad \frac{\partial N}{\partial x} = 2x$

According to Green's theorem

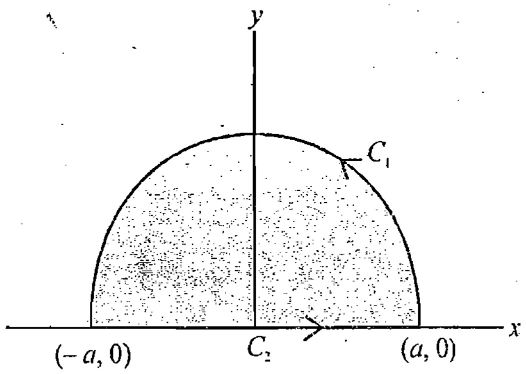


Figure 5.12

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= 2\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (x+y)dxdy$$

$$= 2\iint ydxdy$$

(Since, $\int_{-a}^{a} f(x)dx = 0$ if $f(-x) = f(2y)$ so, $\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdx = 0$)

$$= 2\int_0^a \int_0^\pi r\sin\theta . rd\theta dr \text{ (converting to polar coordinator)}$$

$$= 2\int_0^a r^2[-\cos\theta]_0^\pi dr$$

$$= 4\int_0^a r^2 dr$$

$$= \frac{4a^3}{3}$$

12. Verify Green's theorem in the plane for $\oint (3x^2 - 8y^2)dx + (2y - 3xy)dy$ where C is the boundary of region bounded by $x = 0$ $y = 0$, $x + y = a$.

Solution

The given integral is

$$\oint (3x^2 - 8y^2)dx + (2y - 3xy)dy = \oint Mdx + Ndy$$

Here, $M = 3x^2 - 8y^2, \frac{\partial M}{\partial y} = -16y$

$N = 2y - 3xy, \quad \frac{\partial N}{\partial x} = -3y$

C is a piecewise smooth curve which consists of $C_1 (y = 0)$, $C_2 (x + y = a)$ & $C_3 (x = 0)$ bounding region R as shown in Figure 5.13.

Let us first evaluate the double integral

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = 13\int_0^a \int_0^{a-x} ydydx$$

$$= \frac{13}{2}\int_0^a [y^2]_0^{a-x} dx$$

$$= \frac{13}{2}\int_0^a (a-x)^2\,dx$$

$$= -\frac{13}{6}(a-x)^3\Big|_0^a$$

$$= \frac{13}{6}a^3$$

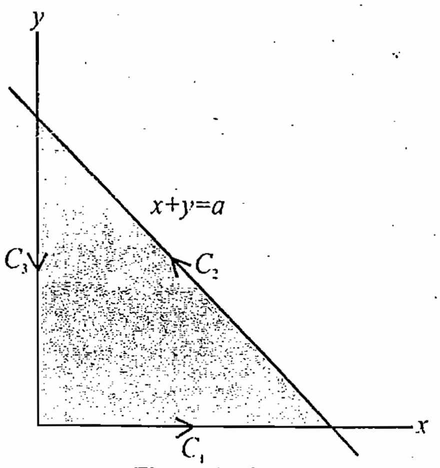


Figure 5.13

Now, let us evaluate the line integral

On C_1, $y = 0$, $dy = 0$, $Mdx + Ndy = 3x^2dx$

x varies from 0 to a.

$$\int_{C_1} Mdx + Ndy = 3\int_0^a x^2\,dx = a^3$$

On C_2, $y = a - x$, $\quad dy = -dx$,

$$Mdx + Ndy = (3x^2 - 8y^2)dx + (2y - 3xy)dy$$
$$= (3x^2 - 8(a-x)^2)dx + (2(a-x) - 3x(a-x))dy$$
$$= (-8x^2 + 19ax + 2x - 8a^2 - 2a)dx$$

x varies from a to 0.

$$\int_{C_2} Mdx + Ndy = \int_a^0 (-8x^2 + 19ax + 2x - 8a^2 - 2a)dx$$

$$= -\frac{8x^3}{3} + \frac{19a}{2}x^2 + x^2 - (8a^2 + 2a)x\Big|_a^0$$

$$= \frac{7}{6}a^3 + a^2$$

On C_3, $x = 0$, $dx = 0$, $Mdx + Ndy = 2ydy$

y varies from a to 0.

$$\int_{C_3} Mdx + Ndy = 2\int_a^0 ydy = -a^2$$

So, $$\oint_C Mdx + Ndy = \int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy + \int_{C_3} Mdx + Ndy$$

$$= a^3 + \left(\frac{7}{6}a^3 + a^2\right) - a^2$$

$$= \frac{13}{6}a^3$$

Hence, $$\oint Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

So, Green' theorem is verified.

13. Evaluate the line integral $\oint_C \frac{xdy - ydx}{x^2 + y^2}$ taken in the positive direction over any closed continuous curve C with the origin inside it.

Solution. The given integral is

$$\oint \frac{xdy - ydx}{x^2 + y^2} = \oint Mdx + Ndy$$

Here, $M=\dfrac{-y}{x^2+y^2}$, $N=\dfrac{x}{x^2+y^2}$

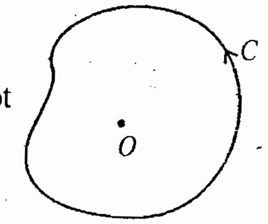


Figure 5.14

Since, M & N are not continuous at origin O. Hence, Green's theorem will not hold good for the given curve C.

Let us enclose the origin by a circle Γ of radius $\in$

Consider the region R enclosed by curve C' made of C, C_2, Γ, C_1.

M and N are continuous function of x and y having continuous partial derivatives

$\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ in R.

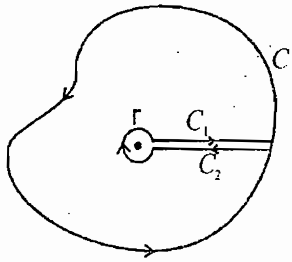


Figure 5.15

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)$$

$$-\frac{(x^2+y^2)-y.2y}{(x^2+y^2)^2}=-\frac{x^2-y^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)=\frac{(x^2+y^2)-x.2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

So. line integral $\oint Mdx+Ndy$ over curve C'.

$$\oint_{C'} Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

$$\Rightarrow \quad \oint_{C'} Mdx+Ndy=\int_C Mdx+Ndy+\int_{C_2} Mdx+Ndy+\int_\Gamma Mdx+Ndy+\int_{C_1} Mdx+Ndy=0$$

But $\quad \oint_{C_1} Mdx+Ndy=-\int_{C_2} Mdx+Ndy$

$$\Rightarrow \quad \oint_{C'} Mdx+Ndy=\int_C Mdx+Ndy+\int_\Gamma Mdx+Ndy=0$$

So, $\quad \displaystyle\int_C Mdx+Ndy=-\int_\Gamma Mdx+Ndy$ ......(1)

In the figure curve Γ is oriented in negative direction.

On the curve Γ, $x=\in\cos\theta \Rightarrow dx=-\in\sin\theta d\theta$

$y=\in\sin\theta\Rightarrow dy=\in\cos\theta d\theta$

θ varies from 2π to 0.

$$\int_\Gamma\frac{xdy-ydx}{x^2+y^2}=\int_{2\pi}^{0}\frac{\in\cos\theta.\in\cos\theta d\theta-\in\sin\theta(-\in\sin\theta)d\theta}{\in^2}$$

$$=\int_{2\pi}^{0}d\theta=-2\pi$$

So, from (1) $\displaystyle\int_C Mdx+Ndy=-\int_\Gamma Mdx+Ndy=2\pi$

14. Using the line integral, compute the area of the loop of Descarte's folium $x^3+y^3=3xy$.

Solution

Putting $y=tx$ in the equation of folium $x^3+y^3=3xy$

$$x = \frac{3t}{1+t^3};\ y = \frac{3t^2}{1+t^3}$$

Let $t = \dfrac{y}{x} = \tan\theta$ where θ varies from 0 to $\pi/2$.

So, t varies from 0 to ∞.

$$dx = \frac{3(1-2t^3)}{(1+t^3)^2}dt$$

$$dy = \frac{3(2t-t^4)}{(1+t^3)^2}dt$$

Area of loop $\quad A = \dfrac{1}{2}\displaystyle\int_C xdy - ydx$

$$= \frac{9}{2}\int_0^\infty \frac{t^2 dt}{(1+t^3)^2} = \frac{3}{2}$$

15. Verify the Green's theorem

$\displaystyle\int_C (1-x^2)ydx + (1+y^2)xdy$ where C is $x^2 + y^2 = 1$

Solution

Here $\displaystyle\oint_C Mdx + Ndy = \oint_C (1-x^2)ydx + (1+y^2)xdy$

So, $M = (1-x^2)y \Rightarrow \dfrac{\partial M}{\partial y} = 1 - x^2$

$N = (1+y^2)x \Rightarrow \dfrac{\partial N}{\partial x} = 1 + y^2$

Let us first evaluate the double integral over region R bounded by the curve C ($x^2 + y^2 = 1$) as shown in Figure 5.16.

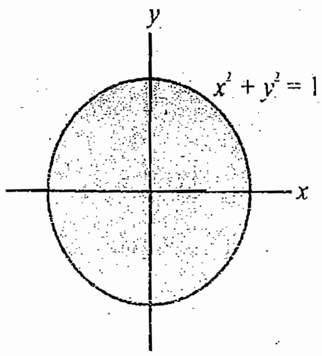


Figure 5.16

$$\iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint (y^2 + x^2)dxdy$$

$$= \int_0^1\int_0^{2\pi} r^3 dr d\theta = \frac{1}{4}\int_0^{2\pi} d\theta = \frac{\pi}{2}$$

Now let us evaluate the line integral $\oint Mdx + Ndy$ as the closed curve C ($x^2 + y^2 = 1$).

On C, $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$y = \sin\theta \Rightarrow dy = \cos\theta\, d\theta$

θ varies from 0 to 2π.

$$\oint Mdx + Ndy = \int_0^{2\pi} (1-\cos^2\theta)\sin\theta(-\sin\theta)d\theta + (1+\sin\theta)\cos\theta\cos\theta d\theta$$

$$= \int_0^{2\pi} (-\sin^2\theta + \cos^2\theta + 2\sin^2\theta\cos^2\theta)d\theta$$

$$= -\int_0^{2\pi}\sin\theta d\theta + \int_0^{2\pi}\cos\theta d\theta + 2\int_0^{2\pi}\sin^2\theta\cos^2\theta d\theta$$

$$= -\pi + \pi + 8\int_0^{\pi/2} \sin^2\theta\cos^2\theta d\theta$$

$$= 8\frac{\overline{|3/2}\,\overline{|3/2}}{2\overline{|3}} = \frac{\pi}{2}$$

Since, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint Mdx + Ndy$

Hence, Green's theorem is verified.

16. Verify Green's theorem in the plane for $\oint_C (xy + x + y)dx + (xy + x - y)dy$ where C is the closed curve $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Solution

Given line integral is

$$\oint_C Mdx + Ndy = \oint (xy + x + y)dx + (xy + x - y)dy$$

$$M = xy + x + y \quad \frac{\partial M}{\partial y} = x + 1$$

$$N = xy + x - y \quad \frac{\partial N}{\partial x} = y + 1$$

Let us first evaluate the double integral over region R bounded by $C = \frac{x^2}{9} + \frac{y^2}{4} = 1$ as shown in Figure 5.17.

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint (y - x)dxdy$$

$$= \int_{-3}^{3} \int_{-\frac{2}{3}\sqrt{9-x^2}}^{\frac{2}{3}\sqrt{9-x^2}} ydydx - \int_{-2}^{2} \int_{-\frac{3}{2}\sqrt{4-y^2}}^{\frac{3}{2}\sqrt{4-y^2}} xdxdy$$

$$= 0 - 0 = 0 \ (\because \int_{-a}^{a} f(x)dx = 0 \text{ if } f(x) \text{ is odd})$$

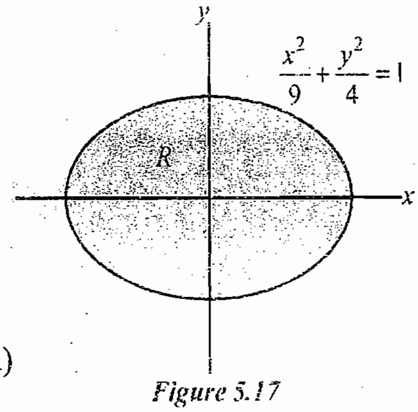


Figure 5.17

Now, let us evaluate the line integral $\oint Mdx + Ndy$ on the curve $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

On C, $x = 3\cos\theta \Rightarrow dx = -3\sin\theta d\theta$

$y = 2\sin\theta \Rightarrow dy = 2\cos\theta d\theta$

θ varies from 0 to 2π.

$$Mdx + Ndy = (xy + x + y)dx + (xy + x - y)dy$$

$$= (6\cos\theta\sin\theta + 3\cos\theta + 2\sin\theta)(-3\sin\theta)d\theta + (6\cos\theta\sin\theta + 3\cos\theta - 2\sin\theta)(2\cos\theta d\theta)$$

$$= (12\cos^2\theta\sin\theta - 18\cos\theta\sin^2\theta - 5\cos\theta\sin\theta - 6\sin^2\theta + 6\cos^2\theta)d\theta$$

So, the line integral

$$\oint M dx + N dy = 12\int_0^{2\pi} \cos^2\theta \sin\theta d\theta + 18\int_0^{2\pi} \cos\theta \sin^2\theta d\theta - 5\int_0^{2\pi} \cos\theta \sin\theta d\theta$$

$$-6\int_0^{2\pi} \sin^2\theta d\theta + 6\int_0^{2\pi} \cos^2\theta d\theta = 0$$

Here, we used $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

$$\int_0^{\pi/2} \cos^2\theta d\theta = \int_0^{\pi/2} \sin^2\theta d\theta = \frac{\pi}{4}$$

17. Verify the Green's theorem

$\oint_C (xy + x + y)dx + (xy + x - y)dy$ where C is the circle $x^2 + y^2 = x$.

Solution.

The given line integral

$$\oint M dx + N dy = \int (xy + x + y)dx + (xy + x - y)dy$$

$$M = (xy + x + y),\ \frac{\partial M}{\partial y} = x + 1$$

$$N = (xy + x - y),\ \frac{\partial N}{\partial x} = y + 1$$

Let us first evaluate the line integral over region R bounded by $C : x^2 + y^2 = x$ as shown in Figure 5.18.

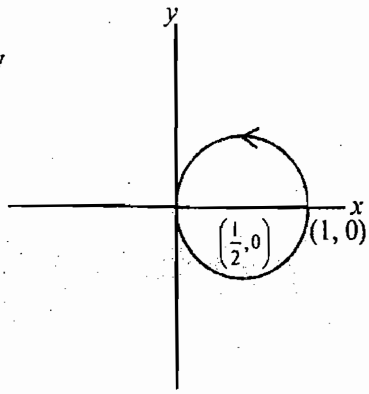


Figure 5.18

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint (y - x)dxdy$$

$$= \int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} y dy dx - \iint x dx dy$$

$$= 0 - \iint r\cos\theta r dr d\theta$$

$$= -\int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} r^2 \cos\theta dr d\theta \text{ (Equation of } C \text{ in polar coordinate is } r = \cos\theta)$$

$$= -\int_{-\pi/2}^{\pi/2} \left.\frac{r^3}{3}\right|_0^{\cos\theta} \cos\theta d\theta$$

$$= -\frac{1}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta$$

$$= -\frac{2}{3}\int_0^{\pi/2} \cos^4\theta d\theta$$

$$= -\frac{2}{3} \cdot \frac{\Gamma(5/2)\Gamma(1/2)}{2\Gamma(3)} = -\frac{\pi}{8}$$

Now, let us evaluate the line integral $\oint Mdx + Ndy$ over the curve

$C : x^2 + y^2 = x \Rightarrow (x-1/2)^2 + y^2 = 1/4.$

On $C, x = \frac{1}{2} + \frac{1}{2}\cos\theta \Rightarrow dx = -\frac{1}{2}\sin\theta d\theta$

$$y = \frac{1}{2}\sin\theta \Rightarrow dy = \frac{1}{2}\cos\theta d\theta$$

$$Mdx + Ndy = (xy + x + y)dx + (xy + x - y)dy$$

$$= \left(\left(\frac{1}{2}+\frac{1}{2}\cos\theta\right)\frac{1}{2}\sin\theta + \frac{1}{2}+\frac{1}{2}\cos\theta + \frac{1}{2}\sin\theta\right)\left(-\frac{1}{2}\sin\theta d\theta\right)$$

$$+\left(\left(\frac{1}{2}+\frac{1}{2}\cos\theta\right)\frac{1}{2}\sin\theta + \frac{1}{2}+\frac{1}{2}\cos\theta - \frac{1}{2}\sin\theta\right)\frac{1}{2}\cos\theta d\theta$$

$$= \left(-\frac{3}{8}\sin^2\theta + \frac{1}{4}\cos^2\theta - \frac{1}{8}\cos\theta\sin^2\theta\right.$$

$$\left. + \frac{1}{8}\cos^2\theta\sin\theta - \frac{1}{8}\cos\theta\sin\theta + \frac{1}{4}\cos\theta - \frac{1}{4}\sin\theta\right)d\theta$$

So, $$\oint_C Mdx + Ndy = -\frac{3}{8}\int_0^{2\pi}\sin^2\theta d\theta + \frac{1}{4}\int_0^{2\pi}\cos^2\theta d\theta - \frac{1}{8}\int_0^{2\pi}\cos\theta\sin^2\theta d\theta$$

$$+\frac{1}{8}\int_0^{2\pi}\cos^2\theta\sin^2\theta d\theta - \frac{1}{8}\int_0^{2\pi}\cos\theta\sin\theta d\theta + \frac{1}{4}\int_0^{2\pi}\cos\theta d\theta - \frac{1}{4}\int_0^{2\pi}\sin\theta d\theta$$

$$= -\frac{\pi}{8}$$

18. Evaluate the line integral $\int_C (yx^3 + xe^y)dx + (xy^3 + ye^y - 2y)dy$ using Green's theorem where C is a circle of radius a.

Solution.

The region enclosed by a circle of radius a as shown in figure.

$$\int_C (yx^3 + xe^y)dx + (xy^3 + ye^y - 2y)dy = \int_C Mdx + Ndy$$

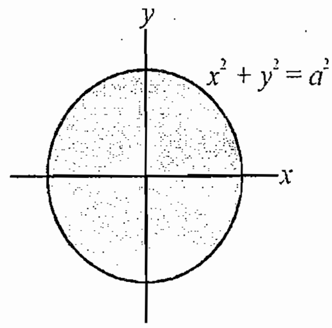


Figure 5.19

Here, $M = yx^3 - xe^y \Rightarrow \frac{\partial M}{\partial y} = x^3 + xe^y$

$$N = xy^3 + ye^y - 2y \Rightarrow \frac{\partial N}{\partial x} = y^3$$

Applying Green's theorem

$$\oint Mdx + Ndy = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$= \iint (y^3 - x^3 - xe^y)dxdy$$

Changing to polar coordinates

$$= \int_0^a\int_0^{2\pi}(r^3\sin^3\theta - r^3\cos^3\theta - r\cos\theta e^{r\sin\theta})rd\theta dr$$

$$= \int_0^a r^4\left(\int_0^{2\pi} \sin^3\theta d\theta\right)dr - \int_0^a r^4\left(\int_0^{2\pi} \cos^3\theta d\theta\right)dr - \int_0^a r\left(\int_0^{2\pi} e^{r\sin\theta} r\cos\theta d\theta\right)dr$$

$$= 0 - 0 - \int_0^a r\left[e^{r\sin\theta}\right]_0^{2\pi} dr = 0$$

19. Evaluate $\oint_C \frac{xdy - ydx}{x^2+4y^2}$ round the circle $x^2+y^2=a^2$ in the positive direction using Green's theorem.

Solution.

The given line integral is

$$\oint_C \frac{xdy - ydx}{x^2+4y^2} = \oint Mdx + Ndy$$

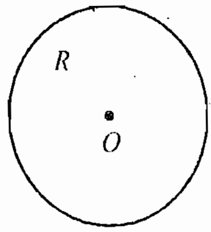


Figure 5.20

Comparing the two integrals,

$$M = -\frac{y}{x^2+4y^2}, \quad \frac{\partial M}{\partial y} = -\frac{x^2-4y^2}{(x^2+4y^2)^2} = \frac{-x^2+4y^2}{(x^2+4y^2)^2}$$

$$N = \frac{x}{x^2+4y^2}, \quad \frac{\partial N}{\partial x} = \frac{-x^2+4y^2}{(x^2+4y^2)^2}$$

The curve C is the circle of radius a. R is the region enclosed by the circle $x^2 + y^2 = a^2$.

M and N are not continuous at origin. So, the Green's theorem will not hold good for the given line integral. Proceeds similarly as done in question (13).

$$\int_C Mdx + Ndy = -\int_\Gamma Mdx + Ndy$$

$$= -\int_\Gamma \frac{xdy - ydx}{x^2+4y^2}$$

$$= -\int_{2\pi}^{0} \frac{\epsilon\cos\theta\, \epsilon\cos\theta - \epsilon\sin\theta(-\epsilon\sin\theta)d\theta}{\epsilon^2\cos^2\theta + 4\epsilon^2\sin^2\theta} \quad \text{(put } x = \epsilon\cos\theta, y = \epsilon\sin\theta\text{)}$$

$$= \int_0^{2\pi} \frac{1}{\cos^2\theta + 4\sin^2\theta} d\theta$$

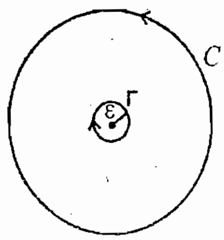


Figure 5.21

$$= \int_0^{2\pi} \frac{\sec^2\theta}{1+4\tan^2\theta} d\theta$$

$$= 2\int_0^{\pi} \frac{\sec^2\theta}{1+4\tan^2\theta} d\theta$$

$$= 4\int_0^{\pi/2} \frac{\sec^2\theta}{1+4\tan^2\theta} d\theta$$

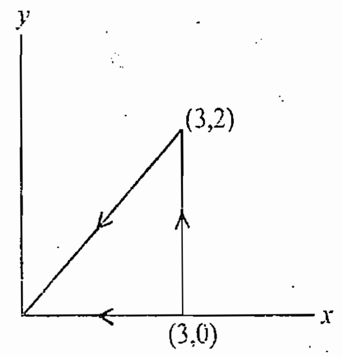
$$= 4\int_0^{\infty} \frac{dt}{1+4t^2}$$

$$= 4\frac{1}{2}.\tan^{-1} 2t\Big|_0^{\infty}$$

$$= \pi$$

EXERCISE

1. Verify Green's theorem in the plane for $\oint_C (x+y)dx+(xy+y)dy$ where C is the square of length 3 on xy plane and its sides parrallel to the x & y axes and one of the vertices at origin.
2. Verify Green's theorem in the plane for $\oint_C (x-2y)dx+(y-2x)dy$ where C is closed curve in xy plane given by $x=\cos t$, $y=2\sin t$ from $t=0$ to $t=2\pi$.
3. Verify Green's theorem in the plane for $\oint_C (x+y^2)dx+(2y-x)dy$ where C is the triangle shown in figure.



4. Verify Green's theorem for $\oint_C (x+y)dx+(x-y)dy$ where C is the closed curve bounded by $y=x^2$ and $y^2=x$.
5. Verify Green's theorem for $\oint_C ydx+xdy$ where C is the circle $x^2+y^2=a^2, z=0$.

History of Zero

The concept of zero as a number and not merely a symbol for separation is attributed to India where by the 9th century AD practical calculations were carried out using zero, which was treated like any other number, even in case of division. The Indian scholar Pingala (circa 5th-2nd century BC) used binary numbers in the form of short and long syllables (the latter equal in length to two short syllables), making it similar to Morse code. He and his contemporary Indian scholars used the Sanskrit word [sunya to refer to zero or void.

Although zero became an integral part of Maya numerals, it did not influence Old World numeral systems.

Quipu, a knotted cord device, used in the Inca Empire and its predecessor societies in the Andean region to record accounting and other digital data, is encoded in a base ten positional system. Zero is represented by the absence of a knot in the appropriate position.

The use of a blank on a counting board to represent 0 dated back in India to 4th century BC.

In China, counting rods were used for decimal calculation since the 4th century BC including the use of blank spaces. Chinese mathematicians understood negative numbers and zero. The Nine Chapters on the Mathematical Art, which was mainly composed in the 1st century AD, stated "[when subtracting] subtract same signed numbers, add differently signed numbers, subtract a positive number from zero to make a negative number, and subtract a negative number from zero to make a positive number.

6

SURFACE INTEGRAL

1. DEFINITION

A integral which is evaluated on a surface is called a surface integral.

Suppose S is a piecewise smooth surface and $\vec{F}(x, y, z)$ is a vector function of position defined and continuous over S.

Let P be any point of the surface S and let $\hat{n}$ be the unit vector at P in the direction of outward drawn normal to the surface S. Then $\vec{F}.\hat{n}$ is a component of $\vec{F}$ normal to surface at point P. The integral of $\vec{F}.\hat{n}$ over the surface is defined as $\int_S \vec{F}\cdot\hat{n}\,dS$.

The surface integral $\int_S \vec{F}\cdot\hat{n}\,dS$ is also called flux of $\vec{F}$ across the surface S. The surface integral is evaluated by reducing it into double integration.

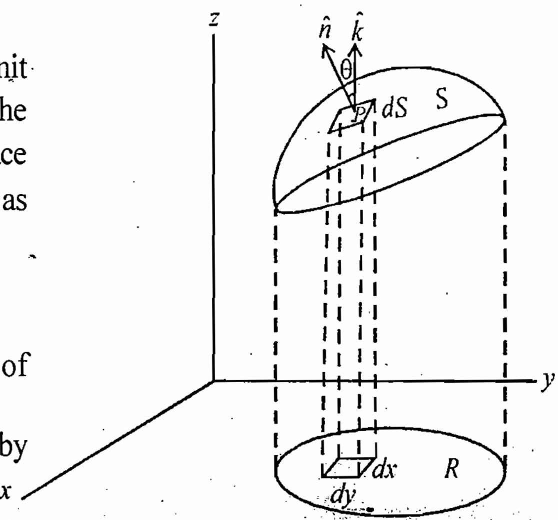


Figure 6.1

The projection of dS on the xy plane is given by $dxdy$.

So, $$dxdy = dS\cos\theta$$

where θ is the angle between $\hat{n}$ and $\hat{k}$ vector

$$\cos\theta = |\hat{n}\cdot\hat{k}|$$

So, $$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|}$$

The unit vector in direction of normal to the surface is given by $\hat{n} = \dfrac{\nabla S}{|\nabla S|}$.

So, $$\vec{F}\cdot\hat{n} = \vec{F}\cdot\frac{\nabla S}{|\nabla S|}$$

Hence, $$\int_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\frac{dxdy}{|\hat{n}\cdot\hat{k}|}$$

The region of integration R for the double integration is given by projection of S over xy–plane as shown in fig. 6.1.

Similarly, $$dS = \frac{dydz}{|\hat{n}\cdot\hat{i}|}$$

So, $\int_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\frac{dydz}{|\hat{n}\cdot\hat{i}|}$

Also, $dS = \frac{dxdz}{|\hat{n}\cdot\hat{j}|}$

So, $\int_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\frac{dxdz}{|\hat{n}\cdot\hat{j}|}$

1. Evaluate $\int_S \vec{F}\cdot\hat{n}dS$ where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is that part of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ which lies in the first octant.

Solution.

The surface of sphere $x^2 + y^2 + z^2 = a^2$ is shown in figure 6.2.

The sphere belongs to a family of level surface given by

$S = x^2 + y^2 + z^2 = c$

So, the unit vector $\hat{n}$ at any point P is given by

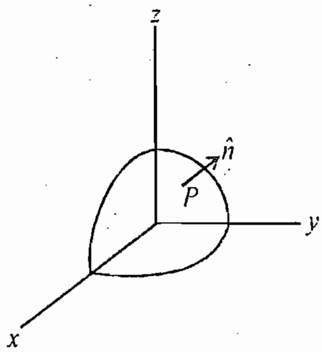


Figure 6.2

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$\hat{n}\cdot\hat{k} = \frac{z}{a}$$

$$\vec{F}\cdot\hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{a}$$

$$= \frac{3xyz}{a}$$

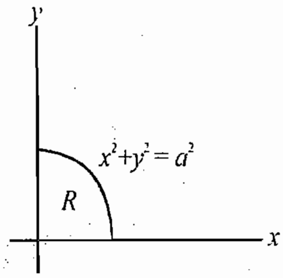


Figure 6.3

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{a}{z}dxdy$$

$$\vec{F}\cdot\hat{n}dS = \frac{3xyz}{a}\cdot\frac{a}{z}\cdot dxdy$$

$$= 3xydxdy$$

$$\int_S \vec{F}\cdot\hat{n}\,dS = 3\iint_R xy\,dx\,dy$$

(The region of integration of double integration given by R)

$$= 3\int_0^{\pi/2}\int_0^a r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= 3\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a \cos\theta\sin\theta d\theta$$

$$= \frac{3a^4}{4}\int_0^{\pi/2}\cos\theta\sin\theta d\theta$$

$$= \frac{3a^4}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{3}{8}a^4$$

2. Evaluate $\int_S \vec{F}\cdot\hat{n}\,dS$ where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = a^2$ along with the bases included in the first octant between $z = 0$ & $z = b$.

Solution.

The cylinder is a piecewise smooth surface consisting of S_1, S_2 and S_3, where S is lower base $z = 0$, S_2 is upper base $z = b$, S_3 is the curved surface of cylinder, as shown in fig. 6.4 & fig. 6.5.

$\hat{n}$ is an outward drawn normal to surface.

$$\int_S \vec{F}\cdot\hat{n}\,dS = \int_{S_1} \vec{F}\cdot\hat{n}\,dS + \int_{S_2} \vec{F}\cdot\hat{n}\,dS + \int_{S_3} \vec{F}\cdot\hat{n}\,dS$$

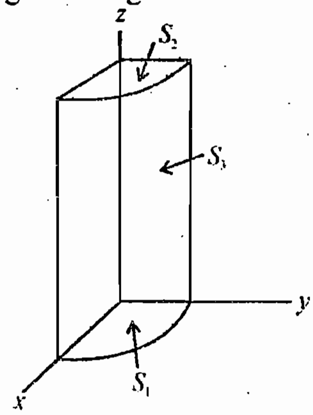


Figure 6.4

On S_1, $\hat{n} = -\hat{k}$, $z = 0$, $dS = dxdy$

$$\vec{F}\cdot\hat{n} = \vec{F}\cdot(-k) = 3y^2z = 0 \text{ (as } z = 0 \text{ on } S_1)$$

So, $$\int_{S_1} \vec{F}\cdot\hat{n}\,dS = 0$$

On S_2, $\hat{n} = \hat{k}$, $z = b$, $dS = dxdy$

$$\vec{F}\cdot\hat{n} = 3y^2z = -3by^2$$

So, $$\int_{S_2} \vec{F}\cdot\hat{n}\,dS = -3b\int\int y^2\,dxdy$$

$$= -3b\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta$$

$$= -3b\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a \sin^2\theta\,d\theta$$

$$= -\frac{3}{4}ba^4\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= -\frac{3}{16}\pi a^4 b$$

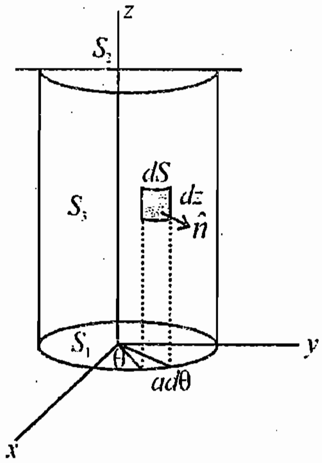


Figure 6.5

The curved surface S_3 belongs to family of level surface $S \equiv x^2 + y^2 = \text{constant}$

The unit normal vector to the surface S_3 is given by

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i} + y\hat{j}}{a}$$

For S_3, $$\vec{F}\cdot\hat{n} = (z\hat{i} + x\hat{j} - 3y^2z\hat{k})\cdot\frac{(x\hat{i} + y\hat{j})}{a}$$

$$= \frac{1}{a}(zx + xy)$$

$$dS = a\,d\theta\,dz$$

On S_3, $x = a\cos\theta$, $y = a\sin\theta$

So, $$\vec{F}\cdot\hat{n} = \frac{1}{a}[az\cos\theta + a^2\sin\theta\cos\theta]$$

$$= z\cos\theta + a\sin\theta\cos\theta$$

The surface integral becomes

$$\int_{S_3} \vec{F}\cdot\hat{n}\,dS = \int_0^b\int_0^{\pi/2}(z\cos\theta + a\sin\theta\cos\theta)\,a\,d\theta\,dz$$

$$= a\int_0^b\left[z\sin\theta + \frac{a}{2}\sin^2\theta\right]_0^{\pi/2} dz$$

$$= a\int_0^b\left(z + \frac{a}{2}\right)dz$$

$$= a\left[\frac{z^2}{2}+\frac{a}{2}z\right]_0^b$$

$$= \frac{ab}{2}(a+b)$$

$$\int_S \vec{F}\cdot\hat{n}\,dS = \int_{S_1}\vec{F}\cdot\hat{n}\,dS+\int_{S_2}\vec{F}\cdot\hat{n}\,dS+\int_{S_3}\vec{F}\cdot\hat{n}\,dS$$

$$= -\frac{3}{18}\pi a^4 b+\frac{ab}{2}(a+b)$$

3. Evaluate $\int_S \vec{F}.\hat{n}dS$, where $\vec{F}=(x+y^2)\hat{i}-2x\hat{j}+2yz\hat{k}$ and S is the surface of the plane $x+2y+3z=6$ in the first octant.

Solution.

The plane belongs to the family of level surface given by $S=x+2y+3z=$ constant.

A unit vector normal to the surface is given by

$$\hat{n} = \frac{\nabla S}{|\nabla S|}=\frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$$

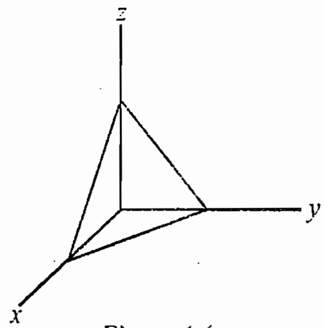


Figure 6.6

$$\vec{F}\cdot\hat{n} = \frac{1}{\sqrt{14}}[(x+y^2)-4x+6yz]$$

$$= \frac{1}{\sqrt{14}}[x+y^2-4x+2y(6-x-2y)]$$

$$\left(z=\frac{1}{3}(6-x-2y)\right)$$

$$= \frac{1}{\sqrt{14}}(12y-3x-3y^2-2xy)$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|}=\frac{\sqrt{14}}{3}\cdot dxdy$$

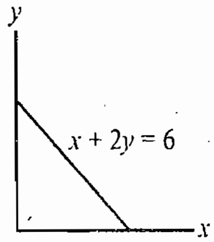


Figure 6.7

$$\vec{F}\cdot\hat{n}dS = \frac{1}{3}(12y-3x-3y^2-2xy)dxdy$$

So, $$\int_S \vec{F}\cdot\hat{n}\,dS = \frac{1}{3}\int_0^6\int_0^{\frac{6-x}{2}}(2y-3x-3y^2-2xy)dydx$$

(The region of double integration is shown in fig. 6.7)

$$= \frac{1}{3}\int_0^6[6y^2-3xy-y^3-xy^2]_0^{\frac{6-x}{2}}\cdot dx$$

$$= \frac{1}{3}\int_0^6\left(-\frac{x^3}{8}+\frac{15}{4}x^2-\frac{45}{2}x+27\right)dx$$

$$= \frac{1}{3}\left[-\frac{x^4}{32}+\frac{5x^3}{4}-\frac{45}{4}x^2+27x\right]_0^6 = 4.5$$

4. Evaluate $\int_S \vec{F}\cdot\hat{n}\,dS$ where $\vec{F}=y\hat{i}+2x\hat{j}-z\hat{k}$ and S is the surface of the plane $2x+y=4$ in the first octant cut off by the plane $z=4$.

Solution.

The surface of the plane $2x + y = 4$ belongs to family of level surface $S = 2x + y =$ constant.

A unit vector normal to the surface

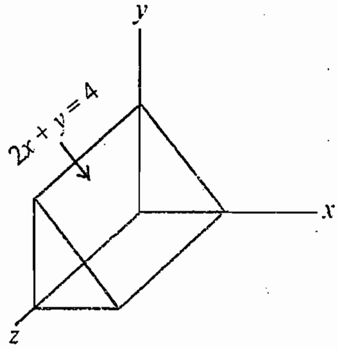


Figure 6.8

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2\hat{i}+\hat{j}}{\sqrt{5}}$$

The integral $$\vec{F}\cdot\hat{n} = (y\hat{i}+2x\hat{j}-z\hat{k})\cdot\left(\frac{2\hat{i}+\hat{j}}{\sqrt{5}}\right)$$

$$= \frac{1}{\sqrt{5}}(2y+2x) = \frac{2}{\sqrt{5}}(x+y)$$

$$\hat{n}\cdot\hat{j} = \frac{1}{\sqrt{5}}(2i+\hat{j})\cdot\hat{j} = \frac{1}{\sqrt{5}}$$

Now, taking projection of the surface on xz plane as shown in fig. 6.9.

$$dS = \frac{dxdz}{|\hat{n}\cdot\hat{j}|} = \sqrt{5}dxdz$$

$$\vec{F}\cdot\hat{n}dS = \frac{2}{\sqrt{5}}(x+y)\sqrt{5}dxdz$$
$$= 2(x+y)\,dxdz$$
$$= 2(x+4-2x)\,dxdz \qquad (y = 4-2x \text{ from the equation of surface})$$
$$= 2(4-x)\,dxdz$$

So, Surface integral becomes

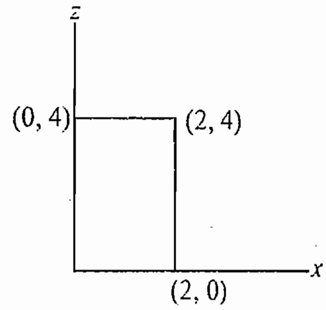


Figure 6.9

$$\int_S \vec{F}\cdot\hat{n}\,dS = 2\int_0^4\int_0^2(4-x)dxdz$$

$$= 2\int_0^4 \left. 4x - \frac{x^2}{2}\right|_0^2 dz$$

$$= 12\int_0^4 dz = 48$$

5. If $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 4x$ in the first octant bounded by the planes $y = 4$ and $z = 6$ then evaluate $\int_S \vec{F}.\hat{n}dS$.

Solution.

The parabolic surface as shown in fig. 6.10 belong to family of level surface $S = 4x - y^2 =$ constant.

The unit normal vector to the parabolic cylinder is given by

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2\hat{i}-y\hat{j}}{\sqrt{y^2+4}}$$

$$\vec{F}\cdot\hat{n} = (2y\hat{i}-z\hat{j}+x^2\hat{k})\cdot\frac{(2\hat{i}-y\hat{j})}{\sqrt{y^2+4}}$$

$$= \frac{4y+yz}{\sqrt{y^2+4}}$$

$$\hat{n}\cdot\hat{i} = \frac{2}{\sqrt{y^2+4}}$$

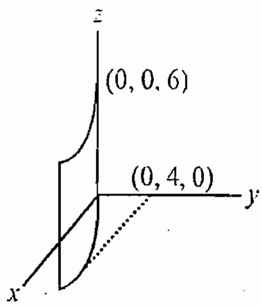


Figure 6.10

$$dS = \frac{dydz}{|\hat{n}\cdot\hat{i}|} = \frac{1}{2}\sqrt{y^2+4}dydz$$

$$\vec{F}\cdot\hat{n}dS = \frac{1}{2}(4y+yz)dydz$$

So, the surface integral reduces to double integral whose region of integration R is given in fig. 6.11

$$\int_S \vec{F}\cdot\hat{n}\,dS = \frac{1}{2}\iint_R (4y+yz)dydz$$

Region R is the projection of parabolic cylinder on yz plane

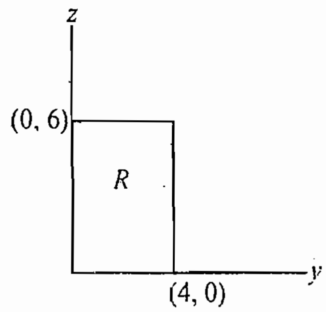


Figure 6.11

$$\int_S \vec{F}\cdot\hat{n}\,dS = \frac{1}{2}\int_0^6\int_0^4 (4y+yz)dydz$$

$$= \frac{1}{2}\int_0^6 \left.2y^2+\frac{y^2z}{2}\right|_0^4 dz$$

$$= \int_0^6 (16+4z)dz$$

$$= 16z+2z^2\Big|_0^6 = 168$$

6. Evaluate $\int_S \vec{F}\cdot\hat{n}\,dS$ over the entire surface of the region above xy plane bounded by the cone $z^2 = x^2+y^2$ and the plane $z=4$ if $\vec{F} = 4xz\hat{i}+xyz^2\hat{j}+3z\hat{k}$.

Solution.

The conical surface S, as shown in the fig. 6.12 belongs to a family of level surface given by

$$S = x^2+y^2-z^2 = \text{constant}.$$

The unit normal vector to cone is given by

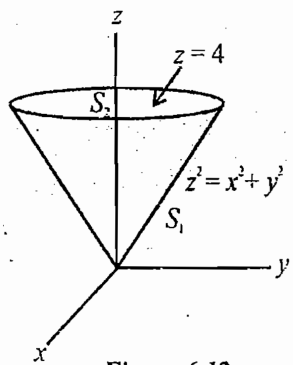


Figure 6.12

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i}+y\hat{j}-z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\vec{F}\cdot\hat{n} = (4xz\hat{i}+xyz^2\hat{j}+3z\hat{k})\cdot\frac{(x\hat{i}+y\hat{j}-z\hat{k})}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{4x^2z+xy^2z^2-3z^2}{\sqrt{x^2+y^2+z^2}}$$

$$\hat{n}\cdot\hat{k} = \frac{-z}{\sqrt{x^2+y^2+z^2}}$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{\sqrt{x^2+y^2+z^2}}{z}dxdy$$

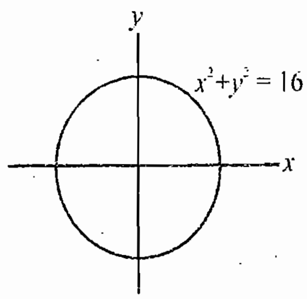


Figure 6.13

$$\vec{F}\cdot\hat{n}dS = \frac{1}{z}(4x^2z+xy^2z^2-3z^2)dxdy$$

$$= (4x^2+xy^2z-3z)dxdy$$

$$= (4x^2+xy^2z-3z)dxdy$$

$$= \left(4x^2+xy^2\sqrt{x^2+y^2}-3\sqrt{x^2+y^2}\right)dxdy$$

So, $$\int_S \vec{F}\cdot\hat{n}\,dS = \iint_R \left(4x^2+xy^2\sqrt{x^2+y^2}-3\sqrt{x^2+y^2}\right)dxdy$$

(R is the region of integration given by projection of cone on xy plane as shown in fig. 6.13)

$$= 4\iint x^2dxdy - 3\iint\sqrt{x^2+y^2}\,dxdy \quad \left(\text{as} \int_{-4}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} xy^2\sqrt{x^2-y^2}\,dxdy = 0\right)$$

$$= 4\int_0^{2\pi}\int_0^4 r^3\cos^2\theta\, drd\theta - 3\int_0^{2\pi}\int_0^4 r^2 drd\theta$$

$$= 4\int_0^{2\pi}\left.\frac{r^4}{4}\right|_0^4 \cos^2\theta\, d\theta - 3\int_0^{2\pi}\left.\frac{r^3}{3}\right|_0^4 d\theta$$

$$= 256\int_0^{2\pi}\cos^2\theta\, d\theta - 64\int_0^{2\pi} d\theta$$

$$= 256\pi - 128\pi = 128\pi$$

On S_2, $\hat{n} = \hat{k}$, $dS = dxdy$

$$\vec{F}\cdot\hat{n} = 3z = 12$$

$$\int_{S_2}\vec{F}\cdot\hat{n}\,dS = 12\iint dxdy$$

$$= 192\pi$$

So, $$\int_S \vec{F}\cdot\hat{n}\,dS = \int_{S_1}\vec{F}\cdot\hat{n}\,dS + \int_{S_2}\vec{F}\cdot\hat{n}\,dS$$

$$= 128\pi + 192\pi = 320\pi$$

7. Evaluate $\oint (x^2+y^2)dS$ where S is the surface of the cone $z^2 = x^2+y^2$ bounded by $z=0$ & $z=3$.

Solution.

Upper part of a cone is given by

$$z = \sqrt{x^2+y^2} \qquad \text{as shown in fig. 6.14.}$$

It belongs to family of level surface given by

$S : \sqrt{x^2+y^2} - z =$ constant.

Outward drawn unit normal vector is given by

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{\frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j} - \hat{k}}{\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}}$$

$$|\hat{n}\cdot\hat{k}| = \frac{1}{\sqrt{2}}$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \sqrt{2}dxdy$$

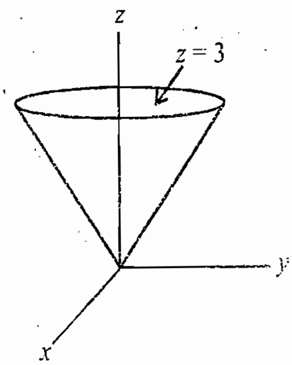


Figure 6.14

S is a piecewise smooth surface consisting of conical part $S_1 : \sqrt{x^2+y^2} - z = 0$ and $S_2 : z = 3$ as shown in fig. 6.14.

On S_1, $dS = \sqrt{2}\,dxdy$

So, $$\int_{S_1}(x^2+y^2)dS = \iint_R (x^2+y^2)\sqrt{2}dxdy$$

The region of double integration R is projection of cone $x^2 + y^2 = z^2$ on the xy plane as shown in fig. 6.15.

$$= \sqrt{2}\int_0^{2\pi}\int_0^3 r^2 \cdot r\, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{r^4}{4}\bigg|_0^3 d\theta = \frac{81\sqrt{2}}{4}\int_0^{2\pi} d\theta$$

$$= \frac{81\sqrt{2}}{2}\pi$$

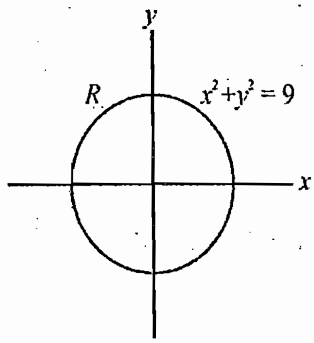


Figure 6.15

On S_2, $z = 3$, $\quad dS = dxdy = rdrd\theta$

$$\int_{S_2}(x^2+y^2)dS = \int_0^{2\pi}\int_0^3 r^2 r d\theta dr$$

$$= \int_0^{2\pi}\frac{r^4}{4}\bigg|_0^3 d\theta$$

$$= \frac{81}{4}\int_0^{2\pi} d\theta$$

$$= \frac{81}{2}\pi$$

So, $$\int_S (x^2+y^2)dS = \int_{S_1}(x^2+y^2)dS + \int_{S_2}(x^2+y^2)dS$$

$$= \frac{81\sqrt{2}}{2}\pi + \frac{81}{2}\pi$$

$$= \frac{81}{2}\pi\left(\sqrt{2}+1\right)$$

8. Evaluate the surface integral $\int_S \frac{dS}{r}$ where S is the portion of the surface of the hyperbolic paraboloid $z = xy$ cut by the cylinder $x^2 + y^2 = 1$ and r is the distance from a point on the surface to z axis.

Solution:

Surface of hyperbolic paraboloid belongs to the family of level surface $S : xy - z = \text{const.}$

The unit normal vector to surface is given by

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{y\hat{i} + x\hat{j} - \hat{k}}{\sqrt{x^2+y^2+1}}$$

$$|\hat{n}\cdot\hat{k}| = \frac{1}{\sqrt{x^2+y^2+1}}$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \sqrt{x^2+y^2+1}\, dxdy$$

So, the surface integral reduces to a double integral

$$I = \int_S \frac{dS}{r} = \iint_R \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}\, dxdy$$

where R is the region of integration of double integral as shown in fig. 6.16 which is projection of surfaces on xy plane.

Changing to polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dxdy = rd\theta dr$$
$$I = \iint_R \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}dxdy$$
$$= \int_0^1\int_0^{2\pi} \sqrt{1+r^2}\, d\theta dr$$
$$= 2\pi\int_0^1 \sqrt{1+r^2}\, dr$$
$$= 2\pi\left[\frac{r}{2}\sqrt{1+r^2}+\frac{1}{2}\log\left(r+\sqrt{1+r^2}\right)\right]_0^1$$
$$= \pi\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]$$

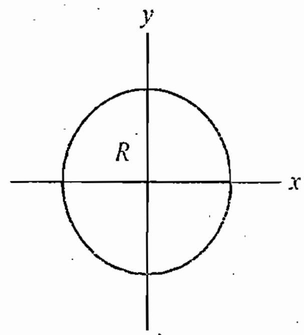


Figure 6.16

9. Evaluate

$$I = \iint xdydz + dzdx + xz^2dxdy$$

where S is the part of sphere $x^2+y^2+z^2=a^2$ in the first octant.

Solution.

S is the part of sphere $x^2+y^2+z^2=a^2$ lying the first octant as shown in fig. 6.17

S belongs to family of level surface given by $S: x^2+y^2+z^2=\text{constant}$

Outward drawn unit normal vector to S,

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$
$$|\hat{n}\cdot\hat{k}| = \frac{z}{a}$$
$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{a}{z}\cdot dxdy$$

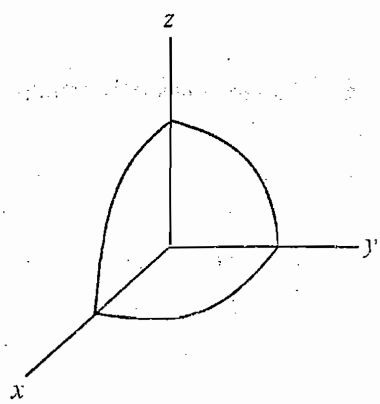


Figure 6.17

The given integral can be written as

$$\iint xdydz + dzdx + xz^2dxdy = \int (x\hat{i}+\hat{j}+xz^2\hat{k})\cdot\hat{n}dS$$
$$= \int \vec{F}\cdot\hat{n}\,dS$$

Where $\vec{F} = x\hat{i}+\hat{j}+xz^2\hat{k}$

$$\vec{F}\cdot\hat{n} = (x\hat{i}+\hat{j}+xz^2\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right)$$
$$= \frac{1}{a}(x^2+y+xz^3)$$
$$\int_S \vec{F}\cdot\hat{n}dS = \iint \frac{(x^2+y+xz^3)}{z}dxdy$$

$$= \iint_R \left[\frac{x^2 + y}{\sqrt{a^2 - x^2 - y^2}} + x(a^2 - x^2 - y^2) \right] dxdy$$

(R is the region of integration as shown in fig. 6.18)

$$= \iint_R \frac{x^2 dxdy}{\sqrt{a^2 - x^2 - y^2}} + \iint_R \frac{y}{\sqrt{a^2 - x^2 - y^2}} dydx + \iint_R x(a^2 - x^2 - y^2)\, dxdy$$

$$= \int_0^a \int_0^{\pi/2} \frac{r^3 \cos^2\theta}{\sqrt{a^2 - r^2}} d\theta dr + \int_0^a \int_0^{\pi/2} \frac{r^2 \sin\theta}{\sqrt{a^2 - r^2}} d\theta dr + \int_0^a \int_0^{\pi/2} r^2(a^2 - r^2)\cos\theta d\theta dr$$

$$= \frac{\pi}{4} \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} dr + \int_0^a \frac{r^2}{\sqrt{a^2 - r^2}} dr + \int_0^a (a^2 r^2 - r^4)\, dr$$

$$= \frac{\pi a^3}{6} + \frac{a^2 \pi}{4} + \left(a^2 \frac{r^3}{3} - \frac{r^5}{5} \right)_0^a$$

$$= \frac{\pi a^3}{6} + \frac{\pi a^2}{4} + \frac{2a^5}{15}$$

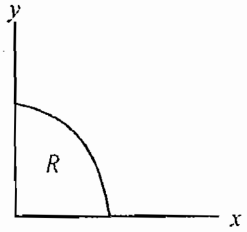


Figure 6.18

10. Evaluate the surface integral $\oint z \cos\theta dS$ over the surface of sphere $x^2 + y^2 + z^2 = a^2$ where θ is the inclination of normal at any point of the sphere with the z axis.

Solution.

S is the surface of sphere consisting of

upper hemisphere $S_1 : z = \sqrt{a^2 - x^2 - y^2}$ and

lower hemisphere $S_2 : z = \sqrt{a^2 - x^2 - y^2}$ as shown in fig. 6.19.

Over S_1, $dS \cos\theta = dxdy$, $z = \sqrt{a^2 - x^2 - y^2}$

$$z\cos\theta dS = \sqrt{a^2 - x^2 - y^2}\, dxdy$$

Over S_2, $dS \cos\theta = dS\cos(\pi - \phi) = -dS\cos\phi = -dxdy$.

$$z = -\sqrt{a^2 - x^2 - y^2}$$

$$z\cos\theta dS = \sqrt{a^2 - x^2 - y^2}\, dxdy$$

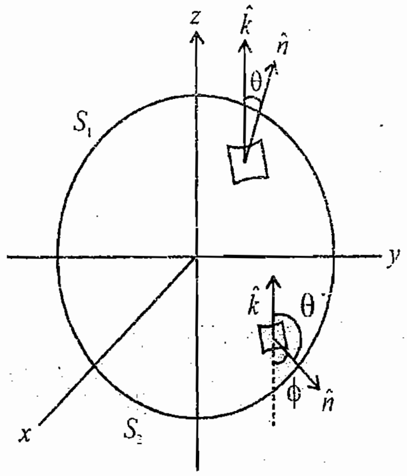


Figure 6.19

Since projection of S_1 and S_2 is same i.e $x^2 + y^2 = a^2$

$$\int_{S_1} z\cos\theta dS = \int_{S_2} z\cos\theta dS$$

So, $$\int_S z\cos\theta dS = \int_{S_1} z\cos\theta\, dS + \int_{S_2} z\cos\theta\, dS$$

$$= 2\iint_R \sqrt{a^2 - x^2 - y^2}\, dxdy$$

(R is the region of integration as shown in fig. 6.20)

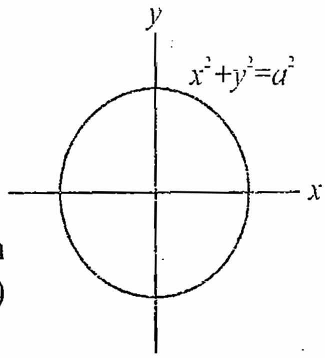


Figure 6.20

$$= 2\int_0^a \int_0^{2\pi} \sqrt{a^2 - r^2}\, r d\theta\, dr$$

$$= 4\pi\int_0^a \sqrt{a^2-r^2}\, r\, dr$$

$$= \frac{4\pi}{3}a^3$$

11. Evaluate $\int_S x\, dS$ where S is the entire surface of solid bounded by the cylinder $x^2+y^2=a^2$ and $z=0$, $z=x+2$.

Solution.

S is piece wise smooth surface consisting of
S_1 : Base of cylinder, $z=0$
S_2 : roof of cylinder, $z= x+2$
S_3 : curved surface of cylinder $x^2+y^2=a^2$
On S_1, $dS = dxdy$

$$\int_{S_1} x\, dS = \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdxdy = 0$$

S_2 belongs to family of level surface given by $S_2: z-x=\text{constant}$.
So, outwards drawn unit normal to S_2

$$\hat{n} = \frac{-\hat{i}+\hat{k}}{\sqrt{2}}$$

On S_2, $dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \sqrt{2}\, dxdy$

So, $\int_{S_2} x\, dS = \sqrt{2}\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdxdy = 0$

On S_3, $dS = ad\theta dz$, $x = a\cos\theta$, $y = a\sin\theta$.
z varies from 0 to $x+2$ i.e. 0 to $2+a\cos\theta$

$$\int_{S_3} x\, dS = \int_0^{2\pi}\int_0^{2+a\cos\theta} a\cos\theta\, a\, dz d\theta$$

$$= a^2\int_0^{2\pi}\cos\theta\cdot(2+a\cos\theta)\, d\theta$$

$$= 2a^2\int_0^{2\pi}\cos\theta d\theta + a^3\int_0^{2\pi}\cos^2\theta d\theta$$

$$= \pi a^3 \qquad \left(\int_0^{2\pi}\cos\theta d\theta = 0\right)$$

So, $\oint_S \vec{F}\cdot\hat{n}\, dS = \int_{S_1}\vec{F}\cdot\hat{n}\, dS + \int_{S_2}\vec{F}\cdot\hat{n}\, dS + \int_{S_3}\vec{F}\cdot\hat{n}\, dS = \pi a^3$

12. Evaluate $\oint \vec{F}\cdot\hat{n}\, dS$ where S is the entire surface of the solid formed by $x^2+y^2=a^2, z=x+1$ and $\hat{n}$ is the outward drawn unit normal and the vector function $\vec{F} = 2x\hat{i} - 3y\hat{j} + z\hat{k}$.

Solution.

S is the piecewise smooth surface consisting of $S_1: z=0$, $S_2: z=x+1$ and $S_3: x^2+y^2=a^2$ (curved surface) as shown in fig. 6.21.

On S_1, $z=0$, $\hat{n}=-\hat{k}, \vec{F}.\hat{n} = -z = 0$

So, $\int_{S_1} \vec{F}.\hat{n}dS = 0$

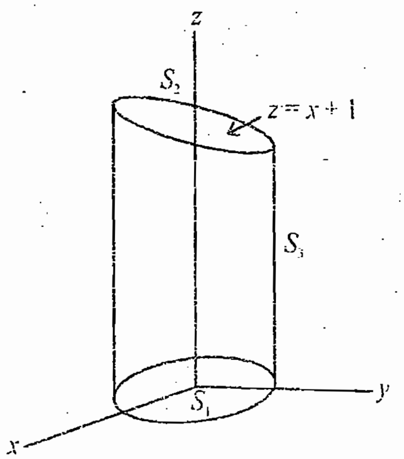


Figure 6.21

On S_2, $z = x + 2$, $\hat{n} = \dfrac{-\hat{i}+\hat{k}}{\sqrt{2}}$ (as done in previous pr...

$$\vec{F}\cdot\hat{n} = \frac{1}{\sqrt{2}}(-2x+z) = \frac{1}{\sqrt{2}}(-x \; 1)$$

$$dS = \frac{dxdy}{|\hat{n}.\hat{k}|} = \sqrt{2}\,dxdy$$

$$\vec{F}.\hat{n}dS = (1-x)\,dxdy$$

$$\int_{S_2} \vec{F}.\hat{n}dS = \iint_R (1-x)\,dxdy$$

$$= \iint dxdy - \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdxdy$$

(R is the region of double integration as shown in fig. 6.22)

$$= \iint dxdy \qquad \left(\text{as } \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdxdy = 0\right)$$

$$= \pi a^2$$

S_3 belong to family of level surface $S_3 : x^2 + y^2 = \text{constant}$.
Outward drawn unit normal vector.

$$\hat{n} = \frac{\nabla S_3}{|\nabla S_3|} = \frac{x\hat{i}+y\hat{j}}{a}$$

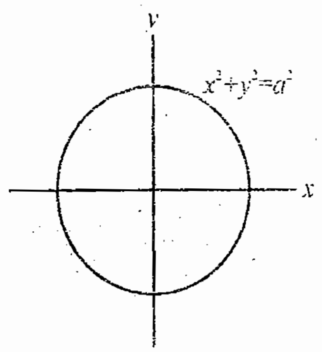


Figure 6.22

On S_3, $\vec{F}\cdot\hat{n} = \dfrac{1}{a}(2x^2 - 3y^2)$, $x = a\cos\theta, y = a\sin\theta$

$$dS = ad\theta\,dz$$

$$\vec{F}\cdot\hat{n}dS = (2x^2 - 3y^2)\,ad\theta\,dz$$

$$= a^3(2\cos^2\theta - 3\sin^2\theta)\,dzd\theta$$

z varies from 0 to $x + 1$, i.e 0 to $1 + a\cos\theta$

$$\int_{S_3} \vec{F}\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^{(1+a\cos\theta)} a^3(2\cos^2\theta - 3\sin^2\theta)\,dzd\theta$$

$$= a^3\int_0^{2\pi}(2\cos^2\theta - 3\sin^2\theta)\,d\theta - a^4\int_0^{2\pi}(2\cos^2\theta - 3\sin^2\theta)\cos\theta\,d\theta$$

$$= -\pi a^3$$

So, $\oint_S \vec{F}\cdot\hat{n}\,dS = \int_{S_1}\vec{F}\cdot\hat{n}\,dS + \int_{S_2}\vec{F}\cdot\hat{n}\,dS + \int_{S_3}\vec{F}\cdot\hat{n}\,dS$

$$= 0 + \pi a^2 - \pi a^3$$

$$= \pi a^2(1-a)$$

13. Evaluate $\int_S xyz\,dS$ over the portion of $x + y + z = a$, $a > 0$, lying in the first octant.

Solution.

S is the surface given by $x + y + z = a$ in the first octant. It belong to family of level surface given by $S : x + y + z = \text{constant}$ as shown in fig. 6.23.

Unit normal vector to the surface S

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{3}}$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \sqrt{3}\,dydx$$

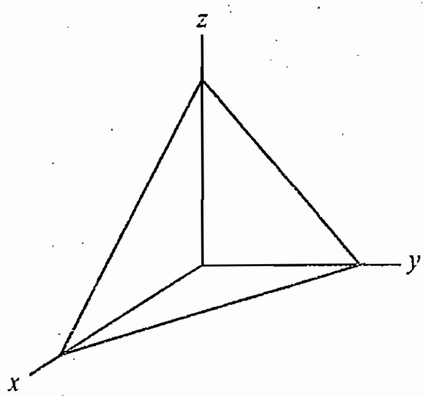


Figure 6.23

So, $$\int_S xyz\,dS = \iint_R xyz\sqrt{3}\,dydx$$

$$= \sqrt{3}\int_0^a\int_0^{a-x} xy(a-x-y)\,dydx$$

(R is the region of double integration as shown in fig. 6.24)

$$= \sqrt{3}\int_0^a \left. x(a-x)\frac{y^2}{2} - \frac{xy^3}{3}\right|_0^{a-x} dx$$

$(z = a-x-y)$

$$= \sqrt{3}\int_0^a \frac{x}{2}(a-x)^3 - \frac{x}{3}(a-x)^3\,dx$$

$$= \frac{\sqrt{3}}{6}\int_0^a x(a-x)^3\,dx = \frac{1}{40\sqrt{3}}a^5$$

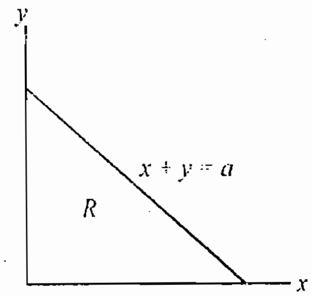


Figure 6.24

14. Evaluate $\int x\,dS$ where S is the portion of the sphere $x^2+y^2+z^2=1$ lying in the first octant.

Solution.

S is the surface of sphere lying in the first octant as shown in fig. 6.25 and belongs to family of level surface $S: x^2+y^2+z^2 = \text{const.}$

An outward drawn unit normal vector to S.

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$\hat{n}\cdot\hat{k} = z$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{1}{z}dxdy = \frac{1}{\sqrt{1-x^2-y^2}}dxdy$$

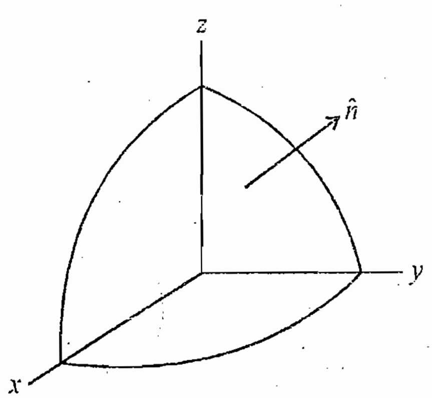


Figure 6.25

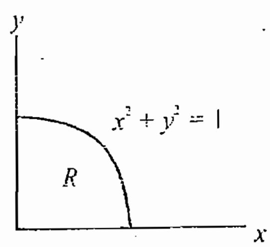


Figure 6.26

$$\int_S x\,dS = \iint_R \frac{x}{\sqrt{1-x^2-y^2}}dxdy$$

(R is the region of double integration as shown in fig. 6.26)

$$= \int_0^1\int_0^{\pi/2} \frac{r^2\cos\theta}{\sqrt{1-r^2}}d\theta dr$$

$$= \int_0^1 \frac{r^2}{\sqrt{1-r^2}}dr = \frac{\pi}{4}$$

15. Evaluate the integral $\int_S \sqrt{1-x^2-y^2}\,dS$ where S is the hemisphere $z=\sqrt{1-x^2-y^2}$.

Solution.

S is the surface of hemisphere $z=\sqrt{1-x^2-y^2}$

An Outward drawn unit normal vector to S

$$\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{dxdy}{z}$$

$$\int_S \sqrt{1-x^2-y^2}\,dS = \iint_R \sqrt{1-x^2-y^2}\frac{dxdy}{z} \quad \left(z=\sqrt{1-x^2-y^2}\right)$$

$$= \iint_R dxdy = \text{Area of region } R$$

(R is the region of double integration as shown in fig. 6.27)

$$= \pi$$

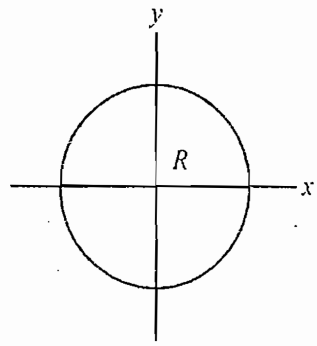


Figure 6.27

16. Evaluate the integral $\int_S x^2y^2dS$ where S is the hemisphere $z=\sqrt{a^2-x^2-y^2}$.

Solution.

S is the surface of hemisphere $z=\sqrt{a^2-x^2-y^2}$.

An outward drawn unit normal vector to S.

$$\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{dxdy}{z}$$

$$\int_S x^2y^2dS = \iint_R \frac{x^2y^2}{z}dxdy$$

(R is the region of double integration as shown in fig. 6.28)

$$= \iint_R \frac{x^2y^2}{\sqrt{a^2-x^2-y^2}}\cdot dxdy$$

$$= \int_0^a\int_0^{2\pi} \frac{r^5\sin^2\theta\cos^2\theta}{\sqrt{a^2-r^2}}d\theta dr$$

$$= 4\int_0^a\int_0^{\pi/2} \frac{r^5}{\sqrt{a^2-r^2}}\sin^2\theta\cos^2\theta\, d\theta dr$$

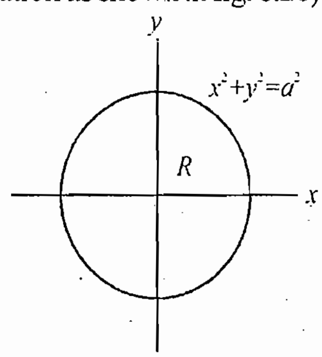


Figure 6.28

$$= 4\int_0^a \frac{r^5}{\sqrt{a^2-r^2}} \frac{\Gamma\frac{3}{2}\Gamma\frac{3}{2}}{2\Gamma 3} dr$$

$$= \frac{\pi}{4}\int_0^a \frac{r^5}{\sqrt{a^2-r^2}} dr$$

$$= \frac{2}{15}\pi a^6$$

17. Evaluate $\int \frac{dS}{r^2}$ where S is the cylinder $x^2+y^2=a^2$ bounded by the planer $z=0$ and $z=b$ and r is the distance between a point on the surface and the origin.

Solution.

S is the surface of cylinder lying between $z=0$ and $z=b$ on S as shown in fig. 6.29, $dS = ad\theta dz$

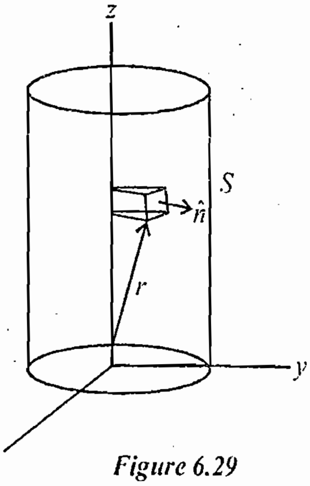


Figure 6.29

$$r = \sqrt{a^2+z^2}$$

$$I = \int \frac{dS}{r^2} = \int_0^b\int_0^{2\pi} \frac{ad\theta dz}{(a^2+z^2)} = 2\pi a\int_0^b \frac{dz}{z^2+a^2}$$

$$= 2\pi \tan^{-1}\frac{z}{a}\Big|_0^b$$

$$= 2\pi \tan^{-1}\frac{b}{a}$$

18. Evaluate $\iint x^3 dydz + y^3 dzdx + z^3 dxdy$ where S is the outer surface of the sphere $x^2+y^2+z^2=a^2$.

Solution.

S is the outer surface of sphere $x^2+y^2+z^2=a^2$ as shown in fig. 6.30. Normal to the outer surface

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

On a surface of sphere,

$$dS = ad\theta \cdot a\sin\theta d\phi$$

$$= a^2 \sin\theta d\theta d\phi$$

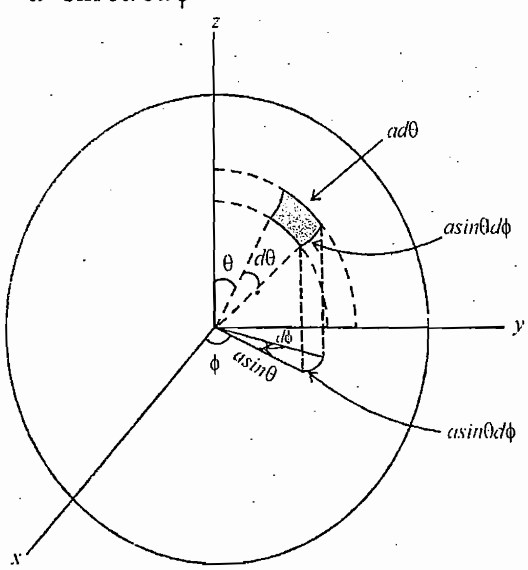


Figure 6.30

$$\iint (x^3dydz + y^3dzdx + z^3dxdy) = \iint (x^3\hat{i} + y^3\hat{j} + z^3\hat{k})\cdot(dy\,dz\hat{i} + dz\,dx\hat{j} + dx\,dy\hat{k})$$

$$= \int (x^3\hat{i} + y^3\hat{j} + z^3\hat{k})\cdot \hat{n}\,dS$$

$$= \iint \frac{(x^4 + y^4 + z^4)}{a} a^2 \sin\theta d\theta d\phi$$

$$= a\iint (x^4 + y^4 + z^4)\sin\theta d\theta d\phi$$

$$= a\int_0^{2\pi}\int_0^{\pi} (a^4 \sin^5\theta\cos^4\phi + a^4\sin^5\theta\sin^4\phi + a^4\cos^4\theta\sin\theta\, d\phi)$$

$$(z = a\cos\theta,\, x = a\sin\theta\cos\phi,\, y = a\sin\theta\sin\phi \text{ on } S)$$

$$\left[\begin{array}{l} \int_0^{\pi} \sin^5\theta d\theta = 2\int_0^{\pi/2} \sin^5\theta d\theta = 2\cdot\frac{\Gamma(3)\Gamma(\frac12)}{2\Gamma(\frac72)} = \frac{16}{15} \\ \int_0^{\pi} \cos^4\theta\sin\theta d\theta = 2\int_0^{\pi/2} \cos^4\theta\sin\theta d\theta = \frac{2\Gamma(\frac52)\Gamma(1)}{2\Gamma(\frac72)} = \frac{2}{5} \end{array}\right]$$

$$= a^5\cdot\frac{16}{15}\cdot\left[\int_0^{2\pi} (\sin^4\phi + \cos^4\phi)d\phi\right] + a^5\cdot\frac{2}{5}\int_0^{2\pi} d\phi$$

$$\left[\begin{array}{l} \int_0^{2\pi} \sin^4\phi\, d\phi = 4\int_0^{\pi/2} \sin^4\phi\, d\phi = 4\cdot\frac{\Gamma(\frac52)\Gamma(\frac12)}{2\cdot\Gamma(3)} = \frac{3}{4}\pi \\ \int_0^{2\pi} \cos^4\phi\, d\phi = 4\int_0^{\pi/2} \cos^4\phi\, d\phi = 4\cdot\frac{\Gamma(\frac52)\Gamma(\frac12)}{2\cdot\Gamma(3)} = \frac{3}{4}\pi \end{array}\right]$$

$$= a^5\cdot\frac{16}{15}\left[\frac{3}{4}\pi + \frac{3}{4}\pi\right] + \frac{2}{5}a^5\cdot 2\pi = \frac{12}{5}\pi a^5$$

19. Evaluate $\iint\limits_S (xz\,dx\,dy + xy\,dy\,dz + yz\,dz\,dx)$ where S is the outer side of the pyramid formed by the planer $x = 0, y = 0, z = 0$ and $x + y + z = a$.

Solution.

S is the piece wise smooth surface formed by

$S_1: x = 0.\ S_2: y = 0,\ S_3: z = 0,\ S_4 = x + y + z = a$ as shown in fig. 6.31.

$$\iint\limits_S xzdxdy + xydydz + yzdzdx = \iint (xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k})\cdot(dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k})$$

$$= \int\limits_S (xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k})\cdot\hat{n}\,dS$$

$$\vec{F} = xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k}$$

Here, $$\oint\limits_S \vec{F}\cdot\hat{n}\,dS = \int\limits_{S_1} \vec{F}\cdot\hat{n}\,dS + \int\limits_{S_2} \vec{F}\cdot\hat{n}\,dS + \int\limits_{S_3} \vec{F}\cdot\hat{n}\,dS + \int\limits_{S_4} \vec{F}\cdot\hat{n}\,dS$$

On $S_1: x = 0, \hat{n} = -\hat{i}, \vec{F}\cdot\hat{n} = -xy = 0$

$$\int\limits_{S_1} \vec{F}\cdot\hat{n}\,dS = 0$$

On $S_2: y = 0,\ \hat{n} = -\hat{j}, \vec{F}\cdot\hat{n} = -yz = 0$

$$\int\limits_{S_2} \vec{F}\cdot\hat{n}\,dS = 0$$

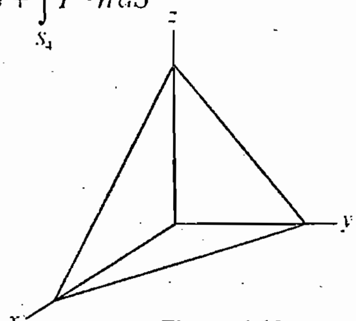


Figure 6.13

On $S_3 : z = 0,\ \hat{n} = -\hat{k}, \vec{F}\cdot\hat{n} = -xz = 0$

$$\int_{S_3} \vec{F}\cdot\hat{n} = 0$$

S_4 belongs to family of level surface

$S_4 : x + y + z =$ constant

$$\hat{n} = \frac{\nabla S_4}{|\nabla S_4|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\vec{F}\cdot\hat{n} = \frac{1}{\sqrt{3}}(xy + yz + zx)$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \sqrt{3}\, dx\, dy$$

$$\int_{S_4} \vec{F}\cdot\hat{n}\, dS = \iint_R (xy + yz + zx)\, dydx$$

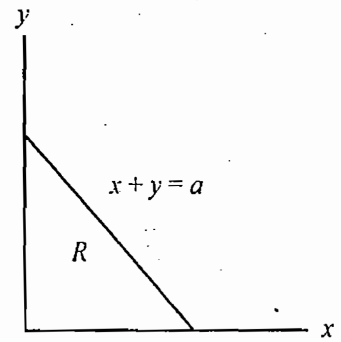


Figure 6.32

(R is the region of integration as shown in fig. 6.32)

$$= \iint \left[xy + (x+y)(a-x-y)\right] dy\, dx$$

$$= \int_0^a \int_0^{a-x} (ax + ay - x^2 - y^2 - xy)\, dydx$$

$$= \int_0^a axy + \frac{ay^2}{2} - x^2y^2 - \frac{y^3}{3} - \frac{xy^3}{2}\Bigg|_0^{a-x} dx$$

$$= \int_0^a a^2x - 2ax^2 + x^3 + \frac{1}{6}(a-x)^3\, dx$$

$$= \frac{1}{8}a^4$$

20. Evaluate the surface integral $\oint (x\hat{i} + y\hat{j} + z\hat{k})\cdot\hat{n}\, dS$ where S is the positive side of the cube formed by the plane $x = 0, y = 0, z = 0$ and $x = 1, y = 1, z = 1$.

Solution.

S is piece wise smooth surface consisting of $S_1 : x = 0, S_2 : y = 0, S_3 : z = 0,\ S_4 : x = 1, S_5 : y = 1,$ $S_6 : z = 1$ as shown in fig. 6.33.

On $S_1 : x = 0,\ dS = dydz,\ \hat{n} = -\hat{i},\ \vec{F}\cdot\hat{n} = -x = 0$

$$\int_{S_1} \vec{F}\cdot\hat{n}\, dS = 0$$

On $S_2 : y = 0,\ dS = dxdz,\ \hat{n} = -\hat{j},\ \vec{F}\cdot\hat{n} = -y = 0$

$$\int_{S_2} \vec{F}\cdot\hat{n}\, dS = 0$$

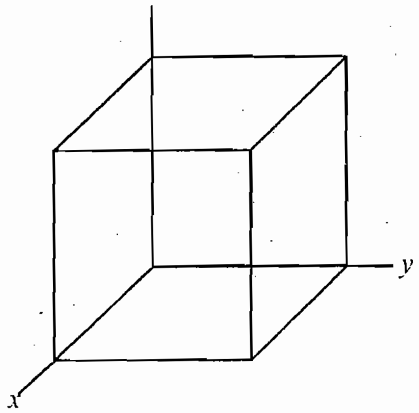


Figure 6.33

On $S_3 : z = 0,\ dS = dxdy,\ \hat{n} = -\hat{k},\ \vec{F}\cdot\hat{n} = -z = 0$

$$\int_{S_3} \vec{F}\cdot\hat{n}\, dS = 0$$

On $S_4 : x = 1,\ dS = dydz,\ \hat{n} = \hat{i},\ \vec{F}\cdot\hat{n} = x = 1$

$$\int_{S_4} \vec{F}\cdot\hat{n}\, dS = \iint dydz = 1$$

On S_5 : $y=1$, $dS = dxdz$, $\hat{n}=\hat{j}$, $\vec{F}\cdot\hat{n}=y=1$

$$\int_{S_5} \vec{F}\cdot\hat{n}\,dS = \iint dxdz = 1$$

On S_6 : $z=1$, $dS = dxdy$, $\hat{n}=\hat{k}$, $\vec{F}\cdot\hat{n}=z=1$

$$\int_{S_6} \vec{F}\cdot\hat{n}\,dS = \iint dxdy = 1$$

So,
$$\oint \vec{F}\cdot\hat{n}dS = \int_{S_1}\vec{F}\cdot\hat{n}dS + \int_{S_2}\vec{F}\cdot\hat{n}dS + \int_{S_3}\vec{F}\cdot\hat{n}dS + \int_{S_4}\vec{F}\cdot\hat{n}dS + \int_{S_5}\vec{F}\cdot\hat{n}dS + \int_{S_6}\vec{F}\cdot\hat{n}dS$$
$$= 3$$

21. Evaluate $\int_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,dS$ where $\cos\alpha, \cos\beta, \cos\gamma$ are directional cosines of the outward drawn normal to the surfaces where S is the outer surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lying above the xy plane.

Solution.

S is the outer surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lying above the xy plane.

An outward drawn unit normal vector to S is given as

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

$$dS = \frac{dydz}{|\hat{n}\cdot\hat{i}|} = \frac{dydz}{\cos\alpha} \Rightarrow dydz = dS\cos\alpha$$

Similarly,
$$dxdy = dS\cos\gamma$$
$$dxdz = dS\cos\beta$$

$$I = \int_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,dS = \iint xdydz + ydxdz + zdxdy$$

$$\left(\iint xdydz = \iint ydxdz = \iint zdxdy = \text{volume of ellipsoid in the above } xy \text{ plane} = \frac{2\pi}{3}abc\right)$$

So, $\int_S (x\cos\beta + y\cos\beta + z\cos\gamma)\,dS = 3\times\frac{2\pi}{3}abc = 2\pi abc$

22. Evaluate $\int_S (x+y+z)(ax+by+cz)\,dS$ where S is the surface of region $x^2+y^2\le 1, 0\le z\le 1$.

Solution.

S is the surface bounding the region $x^2+y^2\le 1\ \&\ 0\le z\le 1$

S is a piecewise smooth surface consisting of

S_1 : lower base $z=0$

S_2 : upper base $z=1$

S_3 : curved surface of cylinder, $x^2+y^2=1$ as shown in fig. 6.34.

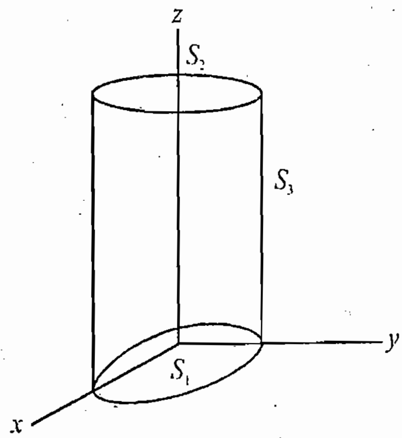


Figure 6.34

On S_1 : $z=0$, $dS = dxdy$

$$\int_{S_1}(x+y+z)(ax+by+cz)\,dS$$
$$= \iint (x+y)(ax+by)\,dxdy$$
$$= \iint (ax^2 + (a+b)xy + by^2)\,dxdy$$

$$= \int_0^{2\pi}\int_0^1 (ar^2\cos^2\theta + (a+b)r^2\sin\theta\cos\theta + br^2\sin^2\theta)r\,dr\,d\theta$$

$$= \int_0^{2\pi} (a\cos^2\theta + (a+b)\sin\theta\cos\theta + b\sin^2\theta)\cdot\left.\frac{r^4}{4}\right|_0^1 d\theta$$

$$= \frac{1}{4}\cdot\int_0^{2\pi} (a\cos^2\theta + b\sin^2\theta + (a+b)\sin\theta\cos\theta\, d\theta$$

$$= \frac{1}{4}a\int_0^{2\pi}\cos^2\theta d\theta + \frac{b}{4}\cdot\int_0^{2\pi}\cos^2\theta d\theta + \frac{(a+b)}{4}\cdot\int_0^{2\pi}\sin\theta\cos\theta d\theta$$

$$= (a+b)\frac{\pi}{4}$$

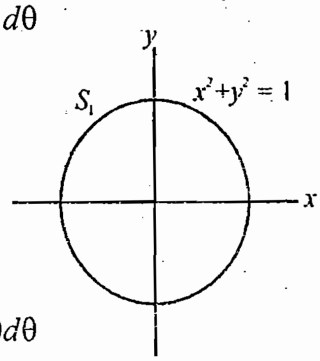


Figure 6.35

On $S_2 : z = 1,\ dS = dxdy$

$$\int_{S_2} (x+y+z)\,(ax+by+cz)\,dS$$

$$= \iint (x+y+z)(ax+by+c)\,dxdy$$

$$= \iint (x+y)(ax+by)\,dxdy + \int_{-1}^{1}\int_{\sqrt{1-y^2}}^{\sqrt{1-y^2}} (a+c)x\,dxdy + \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} (b+c)y\,dydx + c\iint dxdy$$

$$= (a+b)\frac{\pi}{4} + c\pi$$

On $S_3 : x = \cos\theta,\ y = \sin\theta,\ dS = d\theta\, dz,$

$$\int_{S_3} (x+y+z)(ax+by+cz)\,dS$$

$$= \int_0^1\int_0^{2\pi} (\cos\theta + \sin\theta + z)\,(a\cos\theta + b\sin\theta + cz)\,d\theta dz$$

$$= \int_0^1 ((a+b)\pi + 2\pi c z^2)\,dz$$

$$= (a+b)\pi + \frac{2c\pi}{3}$$

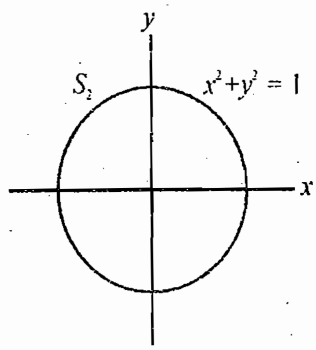


Figure 6.36

$$\int_S (x+y+z)\,(ax+by+cz)\,dS$$

$$= \int_{S_1} + \int_{S_2} + \int_{S_3}$$

$$= (a+b)\frac{\pi}{4} + (a+b)\frac{\pi}{4} + c\pi + (a+b)\pi + \frac{2\pi c}{3}$$

$$= \frac{3}{2}(a+b)\pi + \frac{5c\pi}{3}$$

23. Find the value of surface integral $\iint yzdxdy + xzdydz + xydxdz$ where S is the outer side of the surface formed by the cylinder $x^2 + y^2 = 4$ and the planes $x = 0, y = 0, z = 0$ & $z = 2$.

Solution.

S is a piece wisesmooth surface bounded by $S_1 : x = 0, S_2 : y = 0, S_3 : z = 0$ & $S_4 : x^2 + y^2 = 4$.

$$\iint yzdxdy + xzdydz + xydxdz = \oint_S (xz\hat{i} + xy\hat{j} + yz\hat{k})\cdot\hat{n}dS$$

$$= \oint \vec{F}\cdot\hat{n}dS$$

On S_1, $\hat{n} = -\hat{i}$, $dS = dydz$, $x = 0$, $\vec{F}\cdot\hat{n} = xz = 0$

So, $\int_{S_1} \vec{F}\cdot\hat{n}\,dS = 0$

On S_2, $y = 0$, $\hat{n} = -\hat{j}$, $dS = dxdz$, $\vec{F}\cdot\hat{n} = xy = 0$

So, $\int_{S_2} \vec{F}\cdot\hat{n}\,dS = 0$

On S_3, $z = 0$, $\hat{n} = -\hat{k}$, $dS = dxdy$, $\vec{F}\cdot\hat{n} = yz = 0$

So, $\int_{S_3} \vec{F}\cdot\hat{n}\,dS = 0$

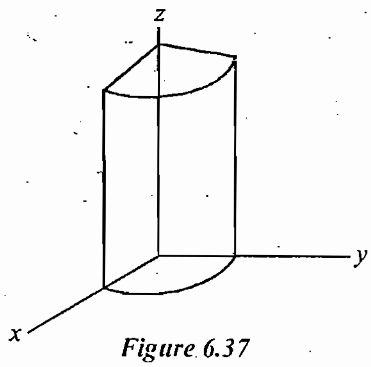


Figure 6.37

On S_4: $x^2+y^2 = 4$, $\hat{n} = \dfrac{x\hat{i}+y\hat{j}}{2}$, $x = 2\cos\theta$, $y = 2\sin\theta$

So, $$\vec{F}\cdot\hat{n} = \frac{x^2z+xy^2}{2} = \frac{4z\cos^2\theta + 8\cos\theta\sin^2\theta}{2}$$

$$= 2z\cos^2\theta + 4\cos\theta\sin^2\theta$$

$$dS = 2\,d\theta dz$$

$$\int_{S_4} \vec{F}\cdot\hat{n}\,dS = \iint (2z\cos^2\theta + 4\cos\theta\sin^2\theta)\,2d\theta dz$$

$$= 4\int_0^2\int_0^{\pi/2} (z\cos^2\theta + 2\cos\theta\sin^2\theta)\,d\theta dz$$

$$= 4\int_0^2 \left(2\cdot\frac{\pi}{4} + \frac{2}{3}\right)dz$$

$$= 4\left[\frac{\pi}{8}z^2 + \frac{2}{3}z\right]_0^2$$

$$= 4\left(\frac{\pi}{2} + \frac{4}{3}\right)$$

So, $$\oint_S \vec{F}\cdot\hat{n}\,dS = 4\left(\frac{\pi}{2} + \frac{4}{3}\right)$$

EXERCISE

1. Evaluate $\int_S \vec{F}.\hat{n}dS$, where $\vec{F} = xy\hat{i} - x^2\hat{j} + (x+z)\hat{k}$, S is the portion of the plane $2x+2y+z = 6$ included in the first octant and $\hat{n}$ is a unit normal to S. **Ans.** $\dfrac{27}{4}$

2. Evaluate $\int_S \vec{A}.\hat{n}dS$ where $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the surface of the plane $2x+3y+6z = 12$ in the first octant. **Ans.** 24

3. Evaluate $\int_S \vec{F}.\hat{n}dS$, where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is that part of the surface of the sphere $x^2+y^2+z^2 = 1$ which lies in the first octant. **Ans.** $\dfrac{3}{8}$

4. Evaluate $\int_S \vec{F}.\hat{n}dS$, where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2+y^2 = 16$ included in the first octant between $z = 0$ to $z = 4$. **Ans.** 64

7

Gauss Divergence Theorem

7.1 GAUSS DIVERGENCE THEOREM

Suppose V is the volume bounded by a closed piecewise smooth surface S. Suppose $\vec{F}(x,y,z)$ is a vector function of position which is continuous and has continuous first partial derivatives in V. Then

$$\oint_S \vec{F}.\hat{n}dS = \int_V \nabla.\vec{F}d\tau$$

where $\hat{n}$ is the outward drawn unit normal vector to S.

In other words, the surface integral of the normal component of a vector $\vec{F}$ over a closed surface is equal to the integral of the divergence of $\vec{F}$ taken over the volume enclosed by the surface.

Proof :

We shall first prove the theorem for a special region V which is bounded by a piecewise smooth closed surface S and has the property that any straight line parallel to any one of the coordinatre axes and intersecting V has only one segment (or a single point) common with V. If R is the orthogonal proejection of S on the xy-plane, then V can be represented in the form

$$f(x,y) \le z \le g(x,y)$$

where (x,y) varies in R.

Obviously $z = g(x,y)$ represents the upper portion of S_1 of S, $z = f(x,y)$ represents the lower portion of S_2 of S and there may be remaining vertical portion of S_3 of S.

We have

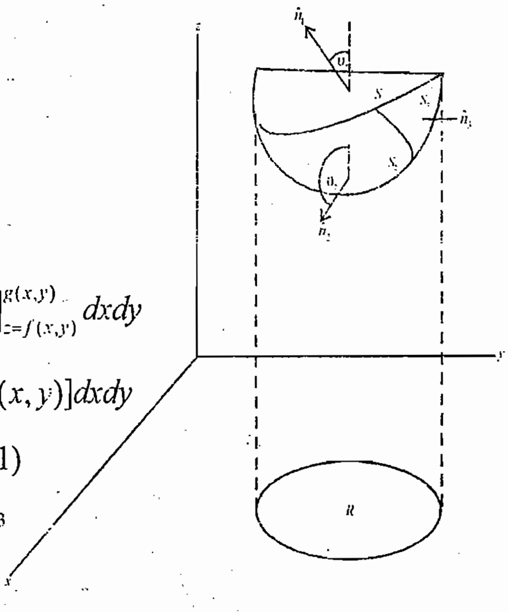
$$\iiint_V \frac{\partial F_3}{\partial z} dV = \iiint_V \frac{\partial F_3}{\partial z} dxdydz$$

$$= \iint_R \left[\int_{z=f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z} dz \right] dxdy$$

$$= \iint_R [F_3(x,y,z)]_{z=f(x,y)}^{g(x,y)} dxdy$$

$$= \iint_R [F_3[x,y,g(x,y)] - F_3[x,y,f(x,y)]]_{z=f(x,y)}^{g(x,y)} dxdy$$

$$= \iint_R F_3[x,y,g(x,y)]dxdy - \iint_R F_3[x,y,f(x,y)]dxdy \quad ....(1)$$



Now for the vertical portion S_3 of S, the normal $\hat{n}_3$ to S_3 makes a right angle θ with $\hat{k}$. Therefore

$$\iint_{S_3} F_3\hat{k}.\hat{n}_3 dS_3 = 0, \text{ since } \hat{k}.\hat{n}_3 = 0$$

For the upper portion of S_1 of S, the normal $\hat{n}_1$ to S_1 makes an acute angle θ_1 with $\hat{k}$. Therefore

$$\hat{k}.\hat{n}_1 dS_1 = \cos\theta_1 dS_1 = dxdy.$$

Hence
$$\iint_{S_1} F_3\hat{k}.\hat{n}_1 dS_1 = \iint_R F_3[x,y,g(x,y)]dxdy$$

For the lower portion of S_2 of S, the normal $\hat{n}_2$ to S_2 makes an obtuse angle θ_2 with $\hat{k}$. Therefore

$$\hat{k}.\hat{n}_2 \ dS_2 = \cos\theta_2 dS_2 = -dxdy.$$

Hence
$$\iint_{S_2} F_3\hat{k}.\hat{n}_2 dS_2 = -\iint_R F_3[x,y,g(x,y)]dxdy$$

$$\therefore \iint_{S_3} F_3\hat{k}.\hat{n}_3 dS_3 + \iint_{S_1} F_3\hat{k}.\hat{n}_1 dS_1 + \iint_{S_2} F_3\hat{k}.\hat{n}_2 dS_2$$

$$= 0 + \iint_R F_3[x,y,g(x,y)]dxdy - \iint_R F_3[x,y,f(x,y)]dxdy$$

or with the help of (1), we get

$$\iint_S F_3\hat{k}.\hat{n}dS = \iiint_V \frac{\partial F_3}{\partial z} dV \qquad \text{....(2)}$$

Similarly, by projecting S on the other co-ordinate planes, we get

$$\iint_S F_3\hat{j}.\hat{n}dS = \iiint_V \frac{\partial F_2}{\partial y} dV \qquad \text{.....(3)}$$

and

$$\iint_S F_1\hat{i}.\hat{n}dS = \iiint_V \frac{\partial F_1}{\partial x} dV \qquad \text{.....(4)}$$

Adding (2), (3) and (4), we get

$$\iint_S (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}).\hat{n}dS = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV$$

or
$$\iiint_V \nabla.\vec{F}dV = \iint_S \vec{F}.\hat{n}dS$$

The proof of the theorem can now be extended to a region V which can be subdivided into fintely many special regions of the above type by drawing auxiliary surfaces. In this case we apply the theorem to each sub-region and then add the results. The sum of the volume integrals over parts of V will be equal to the volume integral over V. The surface integrals over auxiliary surfaces cancel in pairs, while the sum of the remaining surface integrals is equal to the surface integral over the whole boundary S of V.

7.1.1 Deductions from Gauss Divergence Theorem

1. **Green's Theorem.** Let ϕ and ψ are scalar point function which together with their derivatives in any direction are uniform and continuous within the region V bounded by closed surface S then

$$\int_V (\phi\nabla^2\phi - \psi\nabla^2\phi)d\tau = \oint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\hat{n}\, dS$$

Proof: By Gauss Divergence theorem

$$\oint \vec{F}\cdot\hat{n}dS = \int \nabla\cdot\vec{F}d\tau$$

Let
$$\vec{F} = \phi\nabla\psi - \psi\nabla\phi$$

$$\nabla\cdot\vec{F} = \nabla\cdot(\phi\nabla\psi) - \nabla\cdot(\psi\nabla\phi)$$

$$= \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi - \psi\nabla^2\phi - \nabla\psi\cdot\nabla\phi$$
$$= \phi\nabla^2\psi - \psi\nabla^2\phi$$

So, $$\oint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\hat{n}\,dS = \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d\tau \quad ...(1)$$

Since, $$\nabla\psi = \frac{\partial\psi}{\partial n}\hat{n}$$
$$\nabla\phi = \frac{\partial\phi}{\partial n}\hat{n}$$

So, (1) can be written as
$$\oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS = \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d\tau \quad ...(2)$$

Note : Harmonic function : A scalar function ϕ is said to be harmonic function if it satisfies Laplace's equation $\nabla^2\phi = 0$

If ϕ and ψ both are harmonic, i.e. $\nabla^2\phi = \nabla^2\psi = 0$ equation (2) reduces to
$$\oint \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS = 0$$

2. Prove that $\int_V \nabla\phi\,d\tau = \int_S \phi\,\hat{n}\,dS$

Proof : Let $\vec{F} = \phi\vec{C}$ where $\vec{C}$ is any arbitrary constant non zero vector
$$\nabla\cdot\vec{F} = \nabla\phi\cdot\vec{C} + \phi\nabla\cdot\vec{C}$$
$$= \nabla\phi\cdot\vec{C} \quad (\text{as } \nabla.\vec{C} = 0)$$

Applying Divergence theorem
$$\oint_S \vec{F}\cdot\hat{n}\,dS = \int_V \nabla\cdot\vec{F}\,d\tau$$

Where S is bounding surface of V.
$$\oint \phi\vec{C}\cdot\hat{n}dS = \int \nabla.(\phi\vec{C})\,d\tau$$
$$\Rightarrow \quad \vec{C}\cdot\oint\phi\,\hat{n}\,dS = \vec{C}\cdot\int\nabla\phi\,d\tau$$
$$\Rightarrow \quad \vec{C}\cdot\left[\int\nabla\phi\,d\tau - \oint\phi\,\hat{n}\,dS\right] = 0$$

Since, $\vec{C}\cdot\left[\int\nabla\phi\,d\tau - \oint\phi\,\hat{n}\,dS\right]$ is zero for any arbitrary non-zero vector $\vec{C}$.

So, $$\int\nabla\phi\,d\tau - \oint\phi\,\hat{n}\,dS = 0$$

Hence, $$\int\nabla\phi\,d\tau = \oint\phi\,\hat{n}\,dS$$

3. Prove that $\int\nabla\times\vec{g}\,d\tau = \oint \hat{n}\times\vec{g}\,dS$.

Proof: Let $\vec{F} = \vec{g}\times\vec{C}$ where C is any arbitrary non-zero vector.
$$\nabla\cdot\vec{F} = \nabla\cdot(\vec{g}\times\vec{C}) = \vec{C}\cdot\text{curl }\vec{g} - \vec{g}\cdot\text{curl }\vec{C}$$
$$= \vec{C}\cdot\text{curl }\vec{g} \quad (\because \text{ curl }\vec{C} = 0)$$

Applying Divergence theorem
$$\oint \vec{F}\cdot\hat{n}\,dS = \int\nabla\cdot\vec{F}\,d\tau$$
$$\Rightarrow \quad \oint \vec{g}\times\vec{C}\cdot\hat{n}\,dS = \int\vec{C}\cdot\text{curl }\vec{g}\,d\tau$$
$$\Rightarrow \quad \oint(\hat{n}\times\vec{g})\cdot\vec{C}\,dS = \int\vec{C}\cdot\text{curl }\vec{g}\,d\tau \quad \left(\because (\vec{A}\times\vec{B}).\vec{C} = (\vec{C}\times\vec{A})\cdot\vec{B}\right)$$

$$\Rightarrow \qquad \vec{C}\cdot\left[\int \text{curl}\ \vec{g}\, d\tau - \oint \hat{n}\times\vec{g}\, dS\right] = 0$$

Since, $\vec{C}\cdot\left[\int \text{curl}\ \vec{g}\, d\tau - \oint \hat{n}\times\vec{g}\, dS\right]$ is zero for any arbitrary non-zero vector $\vec{C}$,

So, $\int \text{curl}\,\vec{g}\, d\tau - \oint \hat{n}\times\vec{g}\, dS = 0$

So, $$\int_V \nabla\times\vec{g}\, d\tau = \oint_S \hat{n}\times\vec{g}\, dS$$

SOLVED EXAMPLES (OBJECTIVE)

1. Let $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, S be the surface of the sphere $x^2 + y^2 + z^2 = 1$ and $\hat{n}$ be the inward unit normal vector to S. Then $\oint_S \vec{F}\cdot\hat{n}\, dS$ is equal to

(a) 4π (b) -4π (c) 8π (d) -8π

Ans. (d)

$$\oint \vec{F}\cdot\hat{n}\, dS = -\oint \vec{F}\cdot n'\, dS$$

Where $\hat{n}'$ is outward drawn unit normal vector to S i.e. $\hat{n} = -\hat{n}'$

$= -\int \nabla.\vec{F}\, d\tau$ (Gauss Divergence theorem)

$= -6\times$ volume of sphere (Since, $\nabla\cdot\vec{F} = 6$)

$= -8\pi$

2. Let S be a closed surface for which $\iint_S \vec{r}\cdot\hat{n}\, d\sigma = 1$. Then the volume enclosed by the surface is

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 3

Ans. (b)

$$\oint \vec{r}\cdot\hat{n}\, dS = 1$$

$\Rightarrow \quad \int \nabla.\vec{r}\, d\tau = 1$ (Using Gauss Divergence theorem)

$\Rightarrow \quad 3\int d\tau = 1$ (Since, $\nabla\cdot\vec{r} = 3$)

Volume $V = \int d\tau = \frac{1}{3}$

3. Let $V = \left\{(x,y,z)\in \boldsymbol{R}^2, \frac{1}{4} \le x^2 + y^2 + z^2 \le 1\right\}$ and $\vec{F} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^2}$ for $(x,y,z)\in V$.

Let $\hat{n}$ denote the outward unit normal vector to the boundary of V and S denotes the part $\left\{(x,y,z)\in \boldsymbol{R}^3; x^2+y^2+z^2 = \frac{1}{4}\right\}$ of the boundary of V. Then $\int_S \vec{F}\cdot\hat{n}\, dS$ is equal to

(a) -8π (b) -4π (c) 4π (d) 8π

Ans. (a)

Outward unit normal to boundary of V.

$$\hat{n} = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{1/2}$$

$$= -2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\int \vec{F}.\hat{n}dS = -2\int \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^2}.(x\hat{i}+y\hat{j}+z\hat{k})dS$$

$$= -2\int \frac{1}{x^2+y^2+z^2}dS$$

$$= -8\int dS \qquad \left(\text{Since, } x^2+y^2+z^2=\frac{1}{4} \text{ on S}\right)$$

$$= -8\times 4\pi\cdot\frac{1}{4} = -8\pi$$

4. The value of the integral $\oint_S \vec{F}.\hat{n}dS$, where $\vec{F}=3x\,\hat{i}+2y\,\hat{j}+z\,\hat{k}$ and S is the closed surface given by the planes $x=0, x=1, y=0, y=2, z=0$ and $z=3$ is

(a) 6 (b) 18 (c) 24 (d) 36

Ans. (d)

By divergence theorem

$$\oint \vec{F}.\hat{n}dS = \int \nabla\cdot\vec{F}d\tau$$

$$= 6\int_0^1\int_0^2\int_0^3 dx\,dy\,dz = 36$$

5. For any closed surface S, the surface integral $\oint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS$ is equal to

(a) 0 (b) $2\vec{F}$ (c) $\vec{F}$ (d) None

Ans. (a)

By divergence theorem

$$\oint \text{curl}\,\vec{F}\cdot\hat{n}dS = \int \text{div}\left(\text{curl}\,\vec{F}\right)d\tau$$

$$= 0 \quad \text{since, } \nabla\cdot\left(\nabla\times\vec{F}\right)=0$$

6. For any closed surface S, the integral $\oint_S \vec{r}\cdot\hat{n}\,dS$ is equal to

(a) V (b) $2V$ (c) $3V$ (d) $4V$

where V is the volume enclosed by S.

Ans. (c)

By Divergence theorem

$$\oint \vec{r}\cdot\hat{n}\,dS = \int \nabla\cdot\vec{r}\,d\tau$$

$$= \int 3\,d\tau \qquad (\nabla\cdot\vec{r}=3)$$

$= 3\times$ volume enclosed by surface S

$= 3V$

7. If $\vec{F}=ax\,\hat{i}+by\,\hat{j}+cz\,\hat{k}$, a,b,c are constants, then the integral $\oint_S \vec{F}\cdot\hat{n}\,dS$, S as a sphere of radius r is equal to

(a) $(a+b+c)\frac{4}{3}\pi r^3$ (b) $abc\frac{4}{3}\pi r^3$

(c) 0 (d) $(a+b+c)^2\frac{4}{3}\pi r^3$

Ans. (a)

By Divergence theorem

$$\oint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$$

Where V is bounding surface of volume S.

Let $\quad \vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$

$$\oint \vec{F}\cdot\hat{n}\,dS = \oint\left(ax\hat{i}+by\hat{j}+cz\hat{k}\right)\cdot\hat{n}\,dS$$

$$= \int \nabla\cdot(ax\hat{i}+by\hat{j}+cz\hat{k})d\tau \qquad \text{(By Gauss divergence theorem)}$$

$$= (a+b+c)\int d\tau$$

$$= (a+b+c) \times \text{volume of sphere of radius } r$$

$$= (a+b+c)\frac{4}{3}\pi r^3$$

8. If $\hat{n}$ is the outward drawn unit normal vector to S then the integral $\int_V div\,\hat{n}\,d\tau$ is equal to

(a) S (b) $2S$ (c) $3S$ (d) $4S$

where S is the bounding surface of volume V.

Ans. (a)

By Divergence theorem

$$\int \nabla\cdot\vec{F}d\tau = \oint \vec{F}\cdot\hat{n}dS$$

So, $\int_V \nabla\cdot\hat{n}d\tau = \oint_S \hat{n}\cdot\hat{n}\,dS = \oint_S dS = S.$

9. Let S be the surface of the cube bounded by $x=-1, y=-1, z=-1, x=1, y=1, z=1$. The integral $\oint \vec{r}\cdot\hat{n}\,dS$ is equal to

(a) 12 (b) 24 (c) 18 (d) 36

Ans. (b)

Using Divergence theorem

$$\oint \vec{r}\cdot\hat{n}\,dS = \int \nabla\cdot\vec{r}d\tau$$

$$= 3\int d\tau$$

$$= 3\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} dxdydz$$

$$= 3\times 8\int_0^1\int_0^1\int_0^1 dxdydz = 24$$

10. S be the surface of sphere $x^2+y^2+z^2=9$. The integral $\iint_S\left[(x+z)dydz+(y+z)dzdx+(x+y)dxdy\right]$ is equal to

(a) 54π (b) 72π (c) 76π (d) 80π

Ans. (b)

The surface element

$$\hat{n}dS = dydz\,\hat{i} + dxdz\,\hat{j} + dxdy\,\hat{k}$$

So, $\iint\left[(x+z)dydz+(y+z)dzdx+(x+y)dxdy\right]$

$$= \iint\left[(x+z)\hat{i}+(y+z)\hat{j}+(x+y)\hat{k}\right]\cdot\left[dydz\,\hat{i}+dzdx\,\hat{j}+dxdy\,\hat{k}\right]$$

$$= \oint(x+z)\hat{i}+(y+z)\hat{j}+(x+y)\hat{k})\cdot\hat{n}\,dS$$

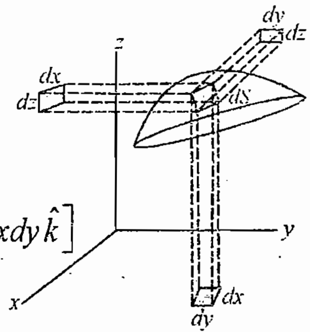


Figure 7.1

$$= \int \nabla \cdot \left((x+z)\hat{i} + (y+z)\hat{j} + (x+y)\hat{k}\right) d\tau$$
$$= \int 2 d\tau$$
$$= 2 \times \frac{4}{3} \cdot \pi (3)^3 = 72\pi$$

11. For any closed surface S, the integral $\oint \hat{n}\, dS$ is equal to

(a) S (b) 0 (c) $2S$ (d) $3S$

Ans. (b)

Let $\vec{C}$ be any arbitrary constant non zero vector
By using divergence theorem

$$\oint_S \vec{C} \cdot \hat{n} dS = \int_V \nabla \cdot \vec{C} d\tau = 0$$

$$\Rightarrow \quad \vec{C} \cdot \oint \hat{n} dS = 0$$

Since, $\vec{C} \cdot \oint \hat{n} dS$ is zero for any arbitrary vector $\vec{C}$

Hence, $\oint \hat{n} dS = 0$

12. For any closed surface S, the integral $\oint \vec{r} \times \hat{n}\, dS$ is equal to

(a) S (b) $2S$ (c) $3S$ (d) 0

Ans. (d)

Let $\vec{C}$ be any arbitrary constant vector

The integral $\vec{C} \cdot \oint \vec{r} \times \hat{n}\, dS$

$$= \oint \vec{C} \cdot (\vec{r} \times \hat{n}) dS$$
$$= \oint (\vec{C} \times \vec{r}) \cdot \hat{n}\, dS$$
$$= \int \nabla \cdot (\vec{C} \times \vec{r}) d\tau \qquad \text{(using divergence theorem)}$$
$$= \int (\vec{r} \cdot \nabla \times \vec{C} - \vec{C} \cdot \nabla \times \vec{r}) d\tau$$
$$= 0 \qquad (\text{Since, } \nabla \times \vec{C} = 0 \ \& \ \nabla \times \vec{r} = 0)$$

Since, $\vec{C} \cdot \oint \vec{r} \times \hat{n}\, dS$ is zero for any arbitrary vector $\vec{C}$

So, $\oint \vec{r} \times \hat{n}\, dS = 0$

13. For any closed surface S, the integral $\oint \nabla\phi \times \hat{n}\, dS$ is equal to

(a) S (b) $\nabla\phi$ (c) 0 (d) None

Ans. (c)

Let $\vec{C}$ be any arbitrary constant vector
The integral

$$\vec{C} \cdot \oint \nabla\phi \times \hat{n}\, dS = \oint \vec{C} \cdot \nabla\phi \times \hat{n}\, dS$$
$$= \oint_S (\vec{C} \times \nabla\phi) \cdot \hat{n}\, dS$$
$$= \int_V \nabla \cdot (\vec{C} \times \nabla\phi) d\tau \qquad \text{(Using divergence theorem)}$$

$$= \int_V \left(\nabla\phi \cdot \text{curl } \vec{C} - \vec{C} \cdot \text{ curl}\nabla\phi\right) d\tau$$

$$= 0 \qquad \text{(as Curl of gradient} = 0)$$

Since, $\vec{C}\cdot\oint \nabla\phi \times \hat{n}\, dS$ is zero for any arbitrary vector $\vec{C}$.

Hence, $\oint \nabla\phi \times \hat{n}\, dS = 0$

14. Let $\vec{a}$ be a constant vector and V is the volume enclosed by the closed surface S. The integral $\oint \hat{n}\times\left(\vec{a}\times\vec{r}\right) dS$ is equal to

(a) $2V\vec{a}$ (b) $V\vec{a}$ (c) 0 (d) $3V\vec{a}$

Ans. (a)

Using $\oint \hat{n}\times\vec{F}dS = \int \nabla\times\vec{F}d\tau$

Putting $\vec{F} = \vec{a}\times\vec{r}$

$$\oint_S \hat{n}\times(\vec{a}\times\vec{r})dS = \int_V \nabla\times(\vec{a}\times\vec{r})d\tau \qquad .....(1)$$

$$\nabla\times(\vec{a}\times\vec{r}) = \sum \hat{i}\times\frac{\partial}{\partial x}(\vec{a}\times\vec{r})$$

$$= \sum \hat{i}\times\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right) = \sum \hat{i}\times(\vec{a}\times\hat{i})$$

$$= \sum (\hat{i}.\hat{i})\vec{a} - (\hat{i}.\vec{a})\hat{i}$$

$$= 3\vec{a} - \vec{a} = 2\vec{a}$$

Equation (1) reduces to

$$\oint_S \hat{n}\times(\vec{a}\times\vec{r})dS = \int_V 2\vec{a}\, d\tau$$

$$= 2\vec{a}\int_V d\tau$$

$$= 2V\vec{a}$$

15. If ϕ is harmonic in V then for any closed surface bounding V, the integral $\oint \frac{\partial\phi}{\partial n} dS$ is equal to

(a) ϕ (b) 2ϕ (c) 0 (d) 4ϕ

Ans. (c)

$$\oint_S \frac{\partial\phi}{\partial n} dS = \oint_S \frac{\partial\phi}{\partial n}\hat{n}\cdot\hat{n}\, dS$$

$$= \oint_S \nabla\phi\cdot\hat{n}\, dS$$

$$= \int_V \nabla\cdot(\nabla\phi)\, d\tau \qquad \text{(Using Gauss divergence theorem)}$$

$$= \int \nabla^2\phi\, d\tau = 0 \qquad (\text{as } \phi \text{ is harmonic i.e. } \nabla^2\phi = 0)$$

16. Let vector $\vec{B}$ is always normal to a given closed surface S. For a region V bounded by S, the integral $\int_V \nabla\times\vec{B}\, d\tau$ is equal to

(a) 0 (b) $\vec{B}$ (c) $2\vec{B}$ (d) $4\vec{B}$

Ans. (a)

We have $\int \nabla \times \vec{B} d\tau = \oint \hat{n} \times \vec{B} dS$

Since, $\vec{B}$ is parallel to normal $\hat{n}$.

So, $\hat{n} \times \vec{B} = 0$

So, $\int \nabla \times \vec{B} d\tau = \oint \hat{n} \times \vec{B} dS = 0$

17. $\vec{F} = (2x+5z)\hat{i} - (x^2z+y)\hat{j} + (y^2+2z)\hat{k}$ the value of integral $\oint_S \vec{F} \cdot \hat{n}\, dS$ where S the surface of sphere having centre at (2, 3, 1) and radius a is equal to

(a) πa^3 (b) $2\pi a^3$ (c) $3\pi a^3$ (d) $4\pi a^3$

Ans. (d)

By Gauss Divergence theorem

$$\oint \vec{F} \cdot \hat{n}\, dS = \int \nabla \cdot \vec{F} d\tau$$

$$= \iiint \nabla \cdot ((2x+5z)\hat{i} - (x^2z+y)\hat{j} + (y^2+2z)\hat{k})\, dxdydz$$

$$= 3\iiint dxdydz$$

$= 3\times$ volume of sphere of radius a

$$= 3\times\frac{4}{3}\pi a^3 = 4\pi a^3$$

18. If S be any closed surface enclosing a volume V and $\vec{F} = 2x\hat{i} + 3y\hat{j} + 7z\hat{k}$. Then, the value of surface integral $\oint \vec{F} \cdot \hat{n}\, dS$ is equal to

(a) $8V$ (b) $10V$ (c) $12V$ (d) $14V$

Ans. (c)

Using Gauss Divergence theorem

$$\oint_S \vec{F} \cdot \hat{n}\, dS = \int_V \nabla \cdot \vec{F} d\tau$$

$$= 12\int_V d\tau$$

$$= 12V$$

19. If S is the surface of sphere $(x-1)^2 + (y-2)^2 + (z-3)^2 = 1$ endosing volume V, and $\vec{F} = x\hat{i} + y\hat{j} + 2z\hat{k}$ then the value of integral $\oint \vec{F} \cdot \hat{n}\, dS$ is equal to

(a) $\frac{8\pi}{3}$ (b) $\frac{11\pi}{3}$ (c) $\frac{14\pi}{3}$ (d) $\frac{16\pi}{3}$

Ans. (d)

V is volume enclosed by sphere S, given by

$(x-1)^2 + (y-2)^2 + (z-3)^2 = 1$

$$\oint_S \vec{F} \cdot \hat{n}\, dS = \int_V \nabla \cdot \vec{F} d\tau$$

$$= 4\int d\tau$$

$= 4\times$ volume of sphere of radius 1

$$= 4\times\frac{4}{3}\pi = \frac{16}{3}\pi$$

SOLVED EXAMPLES (SUBJECTIVE)

1. Prove that

$$\oint_S \phi\vec{A}\cdot\hat{n}\,dS = \int_V \nabla\phi\cdot\vec{A}\,d\tau + \int_V \phi\nabla\cdot\vec{A}\,d\tau$$

where V is volume of region enclosed by closed surface S.

Solution.

By Gauss Divergence theorem

$$\oint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$$

Let $\vec{F} = \phi\vec{A}$

$$\nabla\cdot\vec{F} = \nabla\cdot(\phi\vec{A})$$

$$= \nabla\phi\cdot\vec{A} + \phi\nabla\cdot\vec{A}$$

So, $$\oint_S \phi\vec{A}\cdot\hat{n}\,dS = \int_V \nabla\cdot\left(\phi\vec{A}\right)d\tau$$

$$= \int_V \left(\nabla\phi\cdot\vec{A} + \phi\nabla\cdot\vec{A}\right)d\tau$$

$$= \int_V \nabla\phi\cdot\vec{A}\,d\tau + \int_V \phi\nabla\cdot\vec{A}\,d\tau$$

2. If $\vec{F} = \nabla\phi$ and $\nabla^2\phi = 0$, show that for a closed surfaces $\oint \phi\vec{F}\cdot\hat{n}dS = \int F^2 d\tau$.

Solution.

Using Gauss divergence theorem

$$\oint_S (\phi\vec{F}\cdot\hat{n}\,dS) = \int_V \nabla\cdot(\phi\vec{F})d\tau$$

$$= \int_V (\nabla\phi\cdot\vec{F} + \phi\nabla\cdot\vec{F})d\tau$$

$$= \int_V (\nabla\phi\cdot\nabla\phi + \phi\nabla\cdot\nabla\phi)\,d\tau$$

$$= \int_V (\nabla\phi)^2 d\tau + \int_V \phi\nabla^2\phi\,d\tau$$

$$= \int_V F^2 d\tau + 0 = \int_V F^2\,d\tau$$

3. If ϕ is harmonic in V, then $\oint_S \phi\frac{\partial\phi}{\partial n}dS = \int_V (\nabla\phi)^2\,d\tau$.

Solution.

$$\nabla\phi = \frac{\partial\phi}{\partial n}\hat{n}$$

$$\oint \phi\frac{\partial\phi}{\partial n}dS = \oint \phi\frac{\partial\phi}{\partial n}\hat{n}\cdot\hat{n}\,dS$$

$$= \oint \phi\nabla\phi\cdot\hat{n}\,dS$$

$$= \int \nabla\cdot(\phi\nabla\phi)\,d\tau$$

$$= \int (\nabla\phi\cdot\nabla\phi + \phi\nabla^2\phi)\,d\tau$$

$$= \int (\nabla\phi)^2\,d\tau \qquad (\because \nabla^2\phi = 0)$$

4. Verify divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the cube bounded by $x = 0, y = 0, z = 0, x = a, y = a, z = a.$

Solution.

Let us first find the volume integral

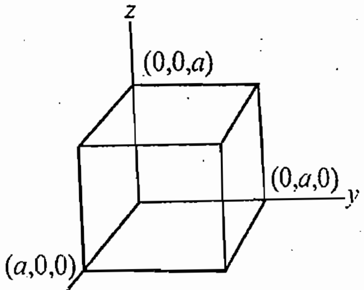


Figure 7.2

$$\int_V \nabla\cdot\vec{F}\,d\tau = \int_0^a\int_0^a\int_0^a (4z-y)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a (4z-y)\left[x\right]_0^a dy\,dz$$

$$= a\int_0^a \left.4yz - \frac{y^2}{2}\right|_0^a dz$$

$$= a^2\int_0^a \left(4z - \frac{a}{2}\right) dz$$

$$= a^2\left[2z^2 - \frac{az}{2}\right]_0^a$$

$$= \frac{3}{2}a^4$$

The region V is bounded by S. S is a piecewise smooth surface consisting of $S_1 (x = 0)$, $S_2 (x = a)$, $S_3 (y = 0)$, $S_4 (y = a)$, $S_5 (z = 0)$, $S_6 (z = a)$.

$$\oint \vec{F}\cdot\hat{n}\,dS = \int_{S_1}\vec{F}\cdot\hat{n}\,dS + \int_{S_2}\vec{F}\cdot\hat{n}\,dS + \int_{S_3}\vec{F}\cdot\hat{n}\,dS + \int_{S_4}\vec{F}\cdot\hat{n}\,dS + \int_{S_5}\vec{F}\cdot\hat{n}\,dS + \int_{S_6}\vec{F}\cdot\hat{n}\,dS \quad .....(1)$$

On S_1, $x = 0$, $\hat{n} = -\hat{i}, \vec{F}\cdot\hat{n} = 0, dS = dydz$

So, $\int_{S_1}\vec{F}\cdot\hat{n}dS = 0$

On S_2, $x = a$, $\hat{n} = \hat{i}, \vec{F}\cdot\hat{n} = 4az,\ dS = dydz$

So, $\int_{S_2}\vec{F}\cdot\hat{n}dS = \int_0^a\int_0^a 4az\,dydz$

$$= \int_0^a 4az\left[y\right]_0^a dz$$

$$= 4a^2\int_0^a zdz = 2a^4$$

On S_3, $y = 0$, $\hat{n} = -\hat{j}$, $dS = dxdz$, $\vec{F}\cdot\hat{n} = 0$

So, $\int_{S_3}\vec{F}\cdot\hat{n}dS = 0$

On S_4, $y = a$, $\hat{n} = \hat{j}$, $\vec{F}\cdot\hat{n} = -a^2$, $dS = dxdz$

So, $\int_{S_4}\vec{F}\cdot\hat{n}dS = -\int_0^a\int_0^a a^2 dxdz$

$$= -a^4$$

On S_5, $z = 0$, $\hat{n} = -\hat{k}$, $\vec{F}\cdot\hat{n} = 0$, $dS = dxdy$

So, $\int_{S_5}\vec{F}\cdot\hat{n}dS = 0$

On S_6, $z=a,\ \hat{n}=\hat{k},\ \vec{F}\cdot\hat{n}=ay,\ dS=dxdy$

$$\int_{S_6}\vec{F}\cdot\hat{n}dS=\int_0^a\int_0^a ay\,dxdy=\frac{a^4}{2}$$

From (1)

$$\oint\vec{F}\cdot\hat{n}\,dS = 0+2a^4+0-a^4+0+\frac{a^4}{2}$$
$$=\frac{3a^4}{2}$$

Hence, $\displaystyle\oint_S\vec{F}\cdot\hat{n}dS=\int_V\nabla\cdot\vec{F}d\tau$

5. Verify divergence theorem for $\vec{F}=(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}$ taken over the rectangular parallelopiped $0\le x\le a, 0\le y\le a, 0\le z\le a$.

Solution.

Let us first calculate the volume integral

$$\vec{F}=(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}$$
$$\nabla\cdot\vec{F}=2(x+y+z)$$

The volume integral

$$\int_V\nabla\cdot\vec{F}d\tau = 2\int_0^a\int_0^a\int_0^a(x+y+z)dxdydz$$
$$=2\int_0^a\int_0^a\left[\frac{x^2}{2}+x(y+z)\right]_0^a dydz$$
$$=2a\int_0^a\int_0^a\left(\frac{a}{2}+(y+z)\right)dydz$$
$$=2a\int_0^a\left.\frac{ay}{2}+\frac{y^2}{2}+zy\right|_0^a dz$$
$$=2a^2\int_0^a\left(\frac{a}{2}+\frac{a}{2}+z\right)dz$$
$$=2a^2\left[az+\frac{z^2}{2}\right]_0^a$$
$$=3a^4$$

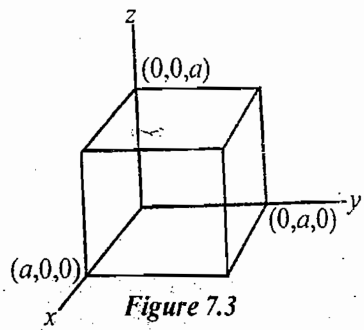


Figure 7.3

The surface S enclosing volume V consists of six pieces of smooth surfaces, $S_1(x=0)$, $S_2(x=a)$, S_3 $(y=0)$, S_4 $(y=a)$, S_5 $(z=0)$, $S_6(z=a)$.

$$\oint_S\vec{F}\cdot\hat{n}\,dS=\int_{S_1}\vec{F}\cdot\hat{n}\,dS+\int_{S_2}\vec{F}\cdot\hat{n}\,dS+\int_{S_3}\vec{F}\cdot\hat{n}\,dS+\int_{S_4}\vec{F}\cdot\hat{n}\,dS+\int_{S_5}\vec{F}\cdot\hat{n}\,dS+\int_{S_6}\vec{F}\cdot\hat{n}\,dS$$

On S_1, $x=0,\ \hat{n}=-\hat{i},\ dS=dydz,\ \vec{F}\cdot\hat{n}=yz$

$$\int_{S_1}\vec{F}\cdot\hat{n}\,dS=\int_0^a\int_0^a yzdydz=\frac{a^4}{4}$$

On S_2, $x=a,\ \hat{n}=\hat{i},\ dS=dydz,\ \vec{F}\cdot\hat{n}=(a^2-yz)$

$$\int_{S_2}\vec{F}\cdot\hat{n}\,dS=\int_0^a\int_0^a(a^2-yz)\,dydz$$

$$= \int_0^a\int_0^a a^2 dydz - \int_0^a\int_0^a yzdydz$$

$$= a^4 - \frac{a^4}{4} = \frac{3}{4}a^4$$

On S_3, $y = 0$, $\hat{n} = -\hat{j}$, $dS = dxdz$, $\vec{F}\cdot\hat{n} = zx$

$$\int_{S_3} \vec{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a zxdxdz = \frac{a^4}{4}$$

On S_4, $y = a$, $\hat{n} = \hat{j}$, $dS = dxdz$, $\vec{F}\cdot\hat{n} = (a^2 - zx)$

$$\int_{S_4} \vec{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a \left(a^2 - zx\right) = \frac{a^4}{4}$$

$$= \frac{3}{4}a^4$$

On S_5, $z = 0$, $\hat{n} = -\hat{k}$, $dS = dxdy$, $\vec{F}\cdot\hat{n} = xy$

$$\int_{S_5} \vec{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a xy\,dxdy = \frac{a^4}{4}$$

On S_6, $z = a$, $\hat{n} = \hat{k}$, $dS = dxdy$, $\vec{F}\cdot\hat{n} = a^2 - xy$

$$\int_{S_6} \vec{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a (a^2 - xy)\,dxdy$$

$$= \frac{3}{4}a^4$$

So, $$\oint \vec{F}\cdot\hat{n}\,dS = \frac{a^4}{4} + \frac{3a^4}{4} + \frac{a^4}{4} + \frac{3a^4}{4} + \frac{a^4}{4} + \frac{3a^4}{4}$$

$$= 3a^4$$

Hence, $$\oint \vec{F}\cdot\hat{n}\,dS = \int \nabla.F.d\tau$$

6. Evaluate

$$\iint x^2dydz + y^2dzdx + 2z(xy - x - y)\,dxdy$$

where S is the surface of the cube

$0 \le x \le a, 0 \le y \le a, 0 \le z \le a$

Solution.

$\hat{n}dS = dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k}$

$x^2dydz + y^2dzdx + 2z(xy - x - y)\,dxdy = \left(x^2\hat{i} + y^2\hat{j} + 2z(xy - x - y)\hat{k}\right)\cdot\hat{n}dS$

So, $$\iint x^2dydz + y^2dzdx + 2z(xy - x - y)dxdy$$

$$= \int_S \left(x^2\hat{i} + y^2\hat{j} + 2z(xy - x - y)\hat{k}\right)\cdot\hat{n}\,dS$$

$$= \int \nabla\cdot\left(x^2\hat{i} + y^2\hat{j} + 2z(xy - x - y)\hat{k}\right)d\tau \quad \text{(By Gauss divergence theorem)}$$

$$= 2\int_0^a\int_0^a xydxdy$$

$$= \frac{a^2}{2}$$

7. Use divergence theorem to evaluate $\iint_S x^3 dydz + x^2 ydzdx + x^2 zdxdy$ where S is the sphere $x^2 + y^2 + z^2 = 1$

Solution.

$$I = \iint x^3 dydz + x^2 ydzdx + x^2 zdxdy$$
$$= \oint (x^3\hat{i} + x^2 y\hat{j} + x^2 z\hat{k}).\hat{n}dS$$
$$= \int \nabla.(x^3\hat{i} + x^2 y\hat{j} + x^2 z\hat{k})d\tau \qquad \text{(By Gauss Divergence theorem)}$$
$$= 5\iiint x^2 dxdydz$$
$$= 5\iiint r^2\sin^2\theta\cos^2\phi.r^2 \sin\theta drd\theta d\phi$$
$$= 5\int_0^1\int_0^{\pi}\int_0^{2\pi} r^4\sin^3\theta\cos^2\phi.d\phi d\theta dr$$
$$= 5\pi\int_0^1\int_0^{\pi} r^4 \sin^3\theta d\theta dr$$
$$= 10\pi\int_0^1\int_0^{\pi/2} r^4 \sin^3\theta d\theta dr$$
$$= 10\pi.\frac{2}{3}\int_0^1 r^4 dr \qquad \left[\int_0^{\pi/2} \sin^3\theta d\theta = \frac{\lceil 2 \lceil \frac{1}{2}}{2\lceil \frac{5}{2}} = \frac{\lceil 2 \lceil \frac{1}{2}}{2.\frac{3}{2}.\frac{1}{2}\lceil \frac{1}{2}} = \frac{2}{3}\right]$$
$$= \frac{4}{3}\pi$$

8. Verify the divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

Solution.

Let us first calculate the volume integral

$$\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$$
$$\nabla.\vec{F} = (4 - 4y + 2z)$$
$$\int_V \nabla.\vec{F}d\tau = \iint\int_0^3 (4-4y+2z)\,dzdydx$$
$$= \iint\left[(4-4y)z + z^2\right]_0^3 dydx$$
$$= \iint (21-12y)\,dydx$$

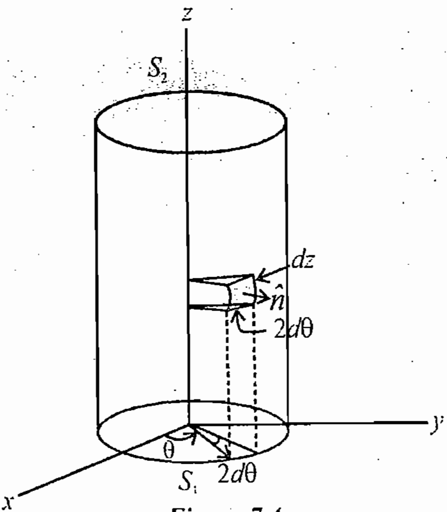


Figure 7.4

The region of double integral is shown in Figure 7.5

$$\iint (21-12y)\,dydx = 21\iint dydx - 12\int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} ydydx$$
$$= 21\iint dydx - 0 \qquad (\because \int_{-a}^{a} f(x)\,dx = 0 \text{ if } f \text{ in odd})$$
$$= 84\pi$$

This volume V is bounded by the surface S which is a piecewise smooth surface consisting of lower base S_1 $(z = 0)$, upper base S_2 $(z = 3)$ and curved surface S_3 $(x^2 + y^2 = 4)$

On S_1, $z=0$, $dS=dxdy$, $\hat{n}=-\hat{k}$, $\vec{F}.\hat{n}=0$

$$\int_{S_1}\vec{F}.\hat{n}dS=0$$

On S_2, $z=3$, $dS=dxdy$, $\hat{n}=\hat{k}$, $\vec{F}.\hat{n}=z^2=9$

$$\int_{S_2}\vec{F}.\hat{n}dS=9\int_{S_2}dS=9\times \text{Area of circle of radius 2}$$
$$=36\pi$$

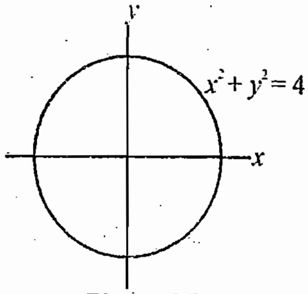


Figure 7.5

On S_3 $x=2\cos\theta$, $y=2\sin\theta$

Equation of S_3 belongs to family of level surface $S: x^2+y^2=$ constant

An outward drawn unit normal vector

$$\hat{n}=\frac{\nabla S}{|\nabla S|}=\frac{x\hat{i}+y\hat{j}}{2}$$

$$\vec{F}.\hat{n}=(2x^2-y^3)$$
$$=8\cos^2\theta-\sin^3\theta$$
$$dS=2d\theta dz$$

$$\int_{S_3}\vec{F}.\hat{n}dS=16\int_0^{2\pi}\int_0^3(\cos^2\theta-\sin^3\theta)dzd\theta$$
$$=48\int_0^{2\pi}(\cos^2\theta-\sin^3\theta)d\theta$$
$$=48\int_0^{2\pi}\cos^2\theta d\theta-48\int_0^{2\pi}\sin^3\theta d\theta \qquad \left(\int_0^{2\pi}\sin^3\theta d\theta=0\right)$$
$$=48\pi$$

The surface integral over S

$$\oint_S\vec{F}.\hat{n}dS=\int_{S_1}\vec{F}.\hat{n}dS+\int_{S_2}\vec{F}.\hat{n}dS+\int_{S_3}\vec{F}.\hat{n}dS$$
$$=0+36\pi+48\pi=84\pi$$

Hence, $$\oint_S\vec{F}.\hat{n}dS=\int_V\nabla.\vec{F}d\tau$$

9. Using divergemce theorem, evaluate. $\oint_S\vec{A}.\hat{n}dS$ where $\vec{A}=x^3\hat{i}+y^3\hat{j}+z^3\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$.

Solution.

Using divergence theorem

$$\oint_S\vec{A}.\hat{n}dS=\int_V\nabla.\vec{A}d\tau$$
$$=3\int(x^2+y^2+z^2)d\tau$$
$$=3\int_0^{2\pi}\int_0^{\pi}\int_0^{a}r^2r^2\sin\theta drd\theta d\phi$$
$$=3\int_0^{2\pi}\int_0^{\pi}\sin\theta\left[\frac{r^5}{5}\right]_0^a d\theta d\phi$$
$$=\frac{3}{5}a^5\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi$$
$$=\frac{3}{5}a^5\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi}d\phi$$

$$= \frac{6}{5}a^5 \int_0^{2\pi} d\phi$$

$$= \frac{12}{5}\pi a^5$$

10. Use divergence theorem to evaluate $\oint \vec{V}.\hat{n}dS$ Where $\vec{V} = x^2z\hat{i} + y\hat{j} - xz^2\hat{k}$ and is the boundary of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4y$.

Solution.

Applying Gauss divergence theorem

$$\oint \vec{V}.\hat{n}dS = \int \nabla.\vec{V}d\tau$$

$$= \int d\tau$$ (This region of volume integration is as shown in Figure. 7.6)

$$= \iint \int_{x^2+y^2}^{4y} dzdxdy$$

$$= \iint (4y - x^2 - y^2)dxdy$$

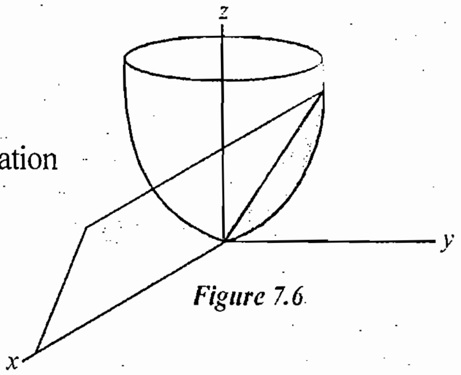


Figure 7.6

The region of integration of double integration in the projection of region V on xy plane as shown in Figure 7.7

$$x^2 + y^2 = 4y$$

$$\Rightarrow \quad x^2 + (y-2)^2 = 4$$

In polar form, $\quad r = 4\sin\theta$

$$I = \iint (4y - x^2 - y^2)dxdy$$

$$= \int_0^{\pi} \int_0^{4\sin\theta} (4r\sin\theta - r^2)rdrd\theta$$

$$= \int_0^{\pi} \left[\frac{4}{3}r^3\sin\theta - \frac{r^4}{4}\right]_0^{4\sin\theta} d\theta$$

$$= \frac{64}{3}\int_0^{\pi} \sin^4\theta d\theta$$

$$= \frac{128}{3}\int_0^{\pi/2} \sin^4\theta d\theta$$

$$= \frac{128}{3}.\frac{3\pi}{16} = 8\pi \qquad \left[\int_0^{\pi/2} \sin^4\theta d\theta = \frac{\Gamma(5/2)\Gamma(1/2)}{2\Gamma(3)} = \frac{\frac{3}{2}.\frac{1}{2}\pi}{2\times2\times1}\right]$$

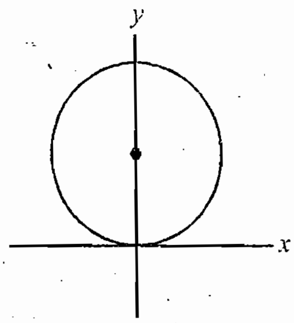


Figure 7.7

11. Verify Gauss divergence theorem for $\vec{F} = xy\hat{i} + z^2\hat{j} + 2yz\hat{k}$ on the tetrahedron $x = y = z = 0, x + y + z = 1$

Solution

Let us find volume integral $\int_V \nabla.\vec{F}d\tau$

V is the region bounded by $x = 0, y = 0, z = 0$ and $x + y + z = 1$ as shown in Figure 7.8

$$\vec{F} = xy\hat{i} + z^2\hat{j} + 2yz\hat{k}$$

$$\nabla.\vec{F} = 3y$$

$$\int \nabla.\vec{F}d\tau = 3\iint\int_0^{1-x-y} ydzdxdy$$

$$= 3\iint_R y(1-x-y)dxdy$$

Where R is the region of double integral obtained by taking projection of V on the xy plane as shown in Figure 7.9

$$= 3\int_0^1\int_0^{1-x}\left(y(1-x)-y^2\right)dydx$$

$$= 3\int_0^1\left[(1-x)\frac{y^2}{2}-\frac{y^3}{3}\right]_0^{1-x}dx$$

$$= \frac{1}{2}\int_0^1(1-x)^3dx$$

$$= -\frac{1}{2}\frac{(1-x)^4}{4}\Bigg|_0^1 = \frac{1}{8}$$

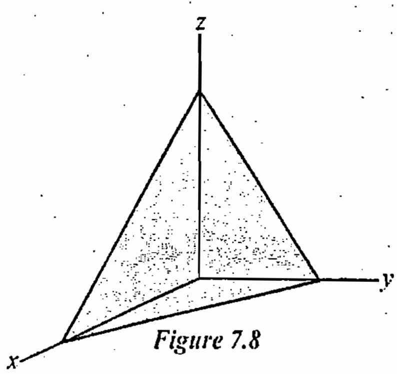


Figure 7.8

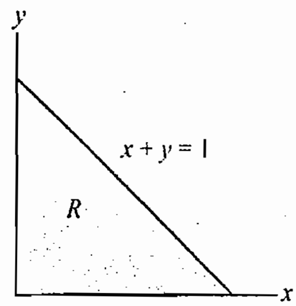


Figure 7.9

The volume V is bounded by surface S. S is a piecewise smooth surface consisting of S_1 $(x=0)$,
$S_2(y=0)$, $S_3(z=0)$, S_4 $(x+y+z=1)$.

On S_1, $x=0$, $\hat{n}=-\hat{i}, dS = dydz, \vec{F}.\hat{n}=0$

$$\int_{S_1}\vec{F}.\hat{n}dS = 0$$

On S_2, $y=0$, $dS = dxdz$, $\hat{n} = -\hat{j}, \vec{F}.\hat{n}=-z^2$

$$\int_{S_2}\vec{F}.\hat{n}dS = \int_0^1\int_0^{1-x} z^2dzdx$$

$$= -\int_0^1\frac{z^3}{3}\Bigg|_0^{(1-x)}dx$$

$$= -\frac{1}{3}\int_0^1(1-x)^3dx$$

$$= \frac{1}{12}(1-x)^4\Bigg|_0^1$$

$$= -\frac{1}{12}$$

On S_3, $z=0$, $dS = dxdy, \hat{n}=-\hat{k}, \vec{F}.\hat{n}=0$

$$\int_{S_3}\vec{F}.\hat{n}dS = 0$$

On S_4, equation of S_4 belongs to family of level surface given by
$S: x+y+z=$ constant

Outward drawn unit normal to S_4

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\vec{F}.\hat{n} = \frac{1}{\sqrt{3}}(xy+z^2+2yz)$$

$$= \frac{1}{\sqrt{3}}(xy+(1-x-y)^2+2y(1-x-y))$$

$$= \frac{1}{\sqrt{3}}(x^2-y^2+xy-2x+1)$$

$$dS = \frac{dxdy}{|\hat{n}.\hat{k}|} = \sqrt{3}\,dxdy$$

So, $$\int_{S_4} \vec{F}.\hat{n}dS = \int_0^1\int_0^{1-x} (x^2 - y^2 + xy - 2x + 1)dydx$$

(The region of double integration is given by projection of V on xy plane as shown in Figure 7.9)

$$= \int_0^1 (x^2 - 2x + 1)y - \frac{y^3}{3} + \frac{xy^2}{2}\Big|_0^{1-x} dx$$

$$= \int_0^1 \left(\frac{2}{3}(1-x)^3 + \frac{x(1-x)^2}{2}\right)dx$$

$$= -\frac{1}{6}(1-x)^4\Big|_0^1 + \frac{1}{2}\left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) = \frac{5}{24}$$

So, $$\oint_S \vec{F}.\hat{n}dS = \int_{S_1} \vec{F}.\hat{n}dS + \int_{S_2} \vec{F}.\hat{n}dS + \int_{S_3} \vec{F}.\hat{n}dS + \int_{S_4} \vec{F}.\hat{n}dS$$

$$= 0 + \left(-\frac{1}{12}\right) + 0 + \frac{5}{24}$$

$$= \frac{1}{8}$$

Hence, $$\oint_S \vec{F}.\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$$

12. Using Divergence theorem, evaluate $I = \iint x^3dydz + x^2ydzdx + x^2zdxdy$ where S is the closed surface bounded by the planes $z = 0$, $z = b$ and the cylinder $x^2 + y^2 = a^2$.

Solution.

$$I = \iint x^3dydz + x^2ydzdx + x^2zdxdy$$

$$= \oint (x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k}).\hat{n}dS$$

$$= \int \nabla\cdot(x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k})d\tau$$

$$= 5\iint\int_0^b x^2dzdxdy$$

$$= 5b\iint_R x^2dxdy$$

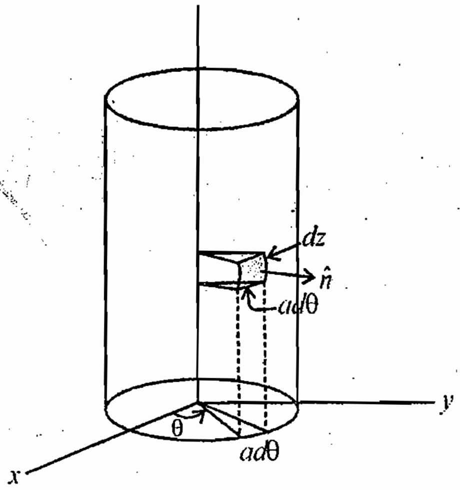


Figure 7.10

(This region of double integral R is given by of projection cylinder on xy plane as shown in Figure 7.11)

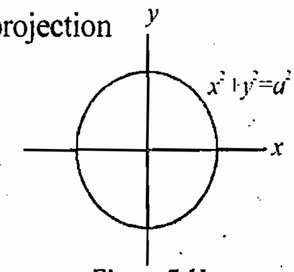


Figure 7.11

$$= 5b\int_0^{2\pi}\int_0^a r^2\cos^2\theta rdrd\theta$$

$$= 5b\int_0^{2\pi} \frac{r^4}{4}\Big|_0^a \cos^2\theta d\theta$$

$$= \frac{5}{4}a^4 b\int_0^{2\pi}\cos^2\theta d\theta$$

$$= \frac{5}{4}\pi a^4 b$$

13. If $\vec{F} = x\hat{i} - y\hat{j} + (z^2-1)\hat{k}$ find the value of $\oint \vec{F}.\hat{n}dS$ where S in the closed surface bounded by the planes $z = 0$, $z = b$ and the cylinder $x^2 + y^2 = a^2$.

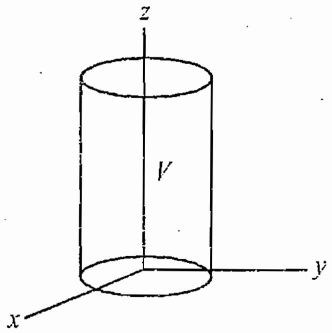


Figure 7.12

Solution.

By Gauss Divergence theorem

$$\oint_S \vec{F}.\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$$

$$\vec{F} = x\hat{i} - y\hat{j} + (z^2-1)\hat{k}$$

$$\nabla\cdot\vec{F} = 2z$$

$$\int \nabla.\vec{F}d\tau = \iint\int_0^b 2z\, dzdxdy$$

$$= \iint z^2\Big|_0^b dxdy$$

$$= b^2\iint_R dxdy$$

(The region of integration R is projection of volume region V on xy plane as shown in Figure 7.13)

$= b^2 \times$ area of circle of radius a

$$= \pi a^2 b^2$$

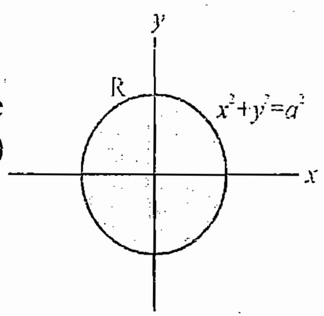


Figure 7.13

14. Evaluate

$$\oint (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}).\hat{n}dS$$

where S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane bounded by this plane.

Solution.

By divergence theorem

$$\oint (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}).\hat{n}dS$$

$$= \int \nabla.(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}).d\tau$$

$$= \iiint 2zy^2 dxdydz$$

$$= \iiint 2r\cos\theta.r^2\sin^2\theta\sin^2\phi.r^2\sin\theta drd\theta d\phi$$

$$= 2\int_0^a\int_0^{\pi/2}\int_0^{2\pi} r^5\sin^3\theta\cos\theta\sin^2\phi\, d\phi\, d\theta\, dr$$

$$= 2\pi\int_0^a\int_0^{\pi/2} r^5\sin^3\theta\cos\theta d\theta dr$$

$$= 2\pi\int_0^a r^5 \frac{\sin^4\theta}{4}\Big|_0^{\pi/2} dr$$

$$= \frac{\pi}{2}\int_0^a r^5 dr = \frac{1}{12}\pi a^6$$

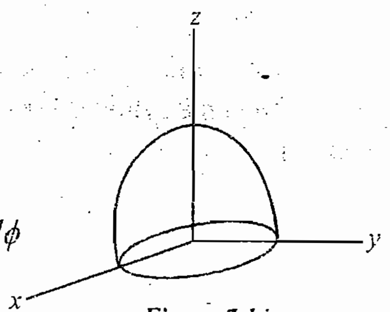


Figure 7.14

15. Evaluate $\oint \vec{F}.\hat{n}dS$ over the entire surface of the region above the xy plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 3$ if $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

Solution.

By Gauss Divergence theorem

$$\oint_s \vec{F}.\hat{n}dS = \int_V \nabla.\vec{F}d\tau$$

$$= \int_V \nabla.(4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k})\,d\tau$$

(V is volume enclosed by cone $z^2 = x^2 + y^2$ and the plane $z = 3$ as shown in Figure 7.15)

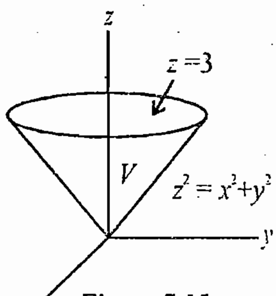


Figure 7.15

$$= \iint \int_{\sqrt{x^2+y^2}}^{3} (4z + xz^2 + 3)\,dzdxdy$$

$$= \iint_R \left. 2z^2 + x\frac{z^3}{3} + 3z \right|_{\sqrt{x^2+y^2}}^{3} dxdy$$

(The region of double integration R is projection of volume V on xy plane as shown Figure 7.16)

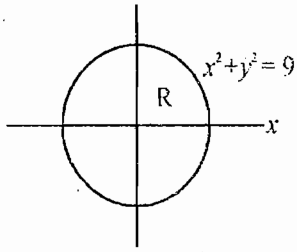


Figure 7.16

$$= \iint [2(9 - x^2 - y^2) + \frac{x}{3}(27 - (x^2 + y^2)^{3/2}) + 3\left(3 - \sqrt{x^2 + y^2}\right)dx\,dy$$

$$= \int_0^3 \int_0^{2\pi} \left[(27 - 2r^2 - 3r) + \frac{1}{3} r\cos\theta(27 - r^3) \right] rd\theta dr$$

$$= 2\pi \int_0^3 (27r - 2r^3 - 3r^2)dr$$

$$= 2\pi \left[\frac{27}{2}r^2 - \frac{r^4}{2} - r^3 \right]_0^3$$

$$= 108\pi$$

16. Evaluate by divergence theorem the integral

$$\iint_3 xz^2 dydz + (x^2 y - z^3)\,dzdx + (2xy + y^2 z)\,dxdy$$

Where S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$

Solution.

The surface is shown in Figure 7.17

$$\hat{n}\,dS = dydz\hat{i} + dxdz\,\hat{j} + dxdy\hat{k}$$

$$= \iint_S xz^2 dydz + (x^2 y - z^3)dzdx + (2xy + y^2 z)dxdy$$

$$= \oint_S (xz^2\hat{i} + (x^2 y - z^3)\,\hat{j} + (2xy + y^2 z)\hat{k})\;\hat{n}dS$$

S is the surface of hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$ as shown in Figure 7.17.

$$\int_V \nabla.(xz^2\,\hat{i} + (x^2 y - z^3)\,\hat{j} + (2xy + y^2 z)\hat{k})d\tau$$

(By Gauss Divergence theorem $\oint_s \vec{F}.\hat{n}dS = \int_V \nabla.Fd\tau$)

$$= \iiint (z^2 + x^2 + y^2)\,dxdydz$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a r^2.r^2 \sin\theta drd\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{r^5}{5}\Bigg|_0^a \sin\theta\, d\theta\, d\phi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi/2} \sin\theta\, d\theta\, d\phi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi/2} d\phi$$

$$= \frac{a^5}{5}\int_0^{2\pi} d\phi$$

$$= \frac{2\pi a^5}{5}$$

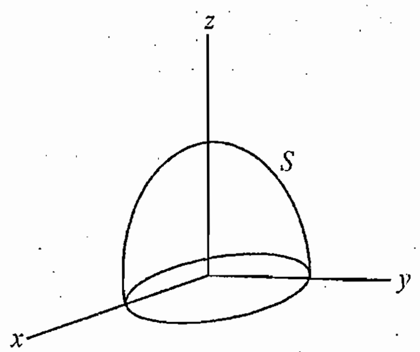


Figure 7.17

17. By using Gauss Divergence theorem,

Evaluate $\oint (x\hat{i} + y\hat{j} + z^2\hat{k})\cdot\hat{n}\, dS$

where S is the closed surface bounded by cone $x^2 + y^2 = z^2$ and the plane $z = 1$.

Solution.

Using Gauss divergence theorem

$$\oint_S \vec{F}\cdot\hat{n}\, dS = \int_V \nabla\cdot\vec{F}\, d\tau$$

$$\int_s (x\hat{i} + y\hat{j} + z^2\hat{k})\cdot\hat{n}\, dS = \int_V \nabla\cdot(x\hat{i} + y\hat{j} + z^2\hat{k})\, d\tau$$

$$= 2\iint\int_{\sqrt{x^2+y^2}}^{1} (z+1)\, dz\, dx\, dy$$

(V is volume enclosed by cone $x^2 + y^2 = z^2$ & the plane $z = 1$ as shown in Figure 7.18)

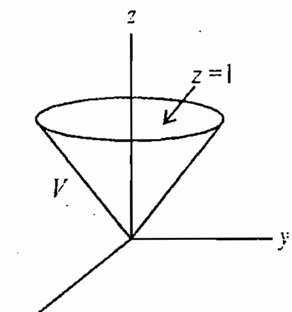


Figure 7.18

$$= 2\iint_R \left.\frac{z^2}{2} + z\right|_{\sqrt{x^2+y^2}}^{1} dx\, dy$$

(The region of integration of double integral R is the projection of volume V on xy plane as shown in Figure 7.19)

$$= \iint (1 - x^2 - y^2) + 2(1 - \sqrt{x^2 + y^2})\, dx\, dy$$

$$= \int_0^{2\pi}\int_0^{1} (3 - 2r - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left.\frac{3}{2}r^2 - \frac{2r^3}{3} - \frac{r^4}{4}\right|_0^1 d\theta = \frac{7}{12}\int_0^{2\pi} d\theta = \frac{7\pi}{6}$$

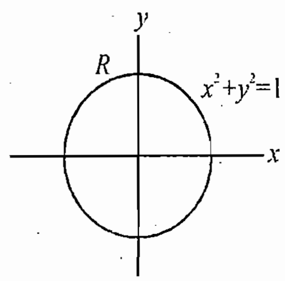


Figure 7.19

18. Verify divergence theorem for $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ & $x = 2$.

Solution.

Let us first find the volume integral $\int_V \nabla\cdot F\, d\tau$, V is the volume enclosed by surface $y^2 + z^2 = 9$ & $x = 2$ in first octant as shown in Figure 7.20.

$$\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$$

$$\nabla\cdot\vec{F} = 4xy - 2y + 8xz$$

$$\int_V \nabla.F d\tau = \iiint_0^2 (4xy-2y+8xz)\,dxdydz$$

$$= \iint_R \left. 2x^2y-2xy+4x^2z \right|_0^2 dydz$$

(R is the projection of V is xy plane as shown in Figure 7.21).

$$= 4\int_0^3\int_0^{\pi/2}(r\cos\theta+4r\sin\theta)\,rd\theta dr$$

$$= 4\int_0^3 r^2\left[\sin\theta-4\cos\theta\right]_0^{\pi/2} dr$$

$$= 20\int_0^3 r^2 dr = 180$$

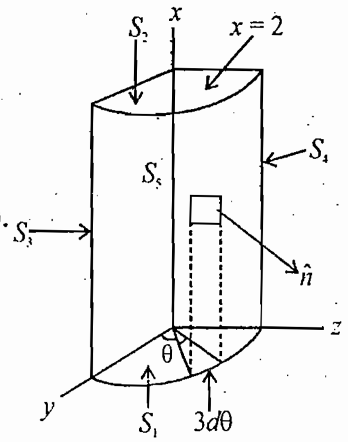


Figure 7.20

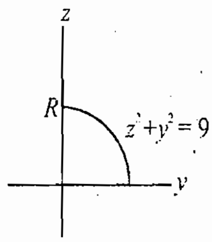


Figure 7.21

Now, let us calculate the surface integral over S. S is a piecewise smooth surface consisting of $S_1(x=0)$, $S_2(x=2)$, $S_3(z=0)$, $S_4(y=0)$, $S_5(y^2+z^2=9)$

On S_1, $x=0$, $dS = dydz$, $\hat{n}=-\hat{i}, \vec{F}.\hat{n}=0$

So, $\quad \int_{S_1}\vec{F}.\hat{n}dS = 0$

On S_2, $x=2$, $dS = dydz$, $\hat{n}=\hat{i}, \vec{F}.\hat{n}=8y$

So, $\quad \int_{S_2}\vec{F}.\hat{n}dS = 8\iint ydydz$

$$= 8\int_0^3\int_0^{\pi/2} r\cos\theta\, rd\theta dr$$

$$= 8\int_0^3 r^2 dr$$

$$= 72$$

On S_3, $z=0$, $dS = dxdy$, $\hat{n}=-\hat{k}, \vec{F}.\hat{n}=0$

So, $\quad \int_{S_3}\vec{F}.\hat{n}dS = 0$

On S_4, $y=0$, $dS = dxdy$, $\hat{n}=-\hat{j}, \vec{F}.\hat{n}=0$

So, $\quad \int_{S_4}\vec{F}.\hat{n}dS = 0$

On S_5, $y^2+z^2=9$, $dS = 3d\theta\, dx$

$$\hat{n}=\frac{y\hat{j}+z\hat{k}}{3}, \vec{F}.\hat{n}=\frac{1}{3}(4xz^3-y^3)$$

let $y=3\cos\theta, z=3\sin\theta$

$$\vec{F}.\hat{n} = 9(4x\sin^3\theta-\cos^3\theta)$$

$$\vec{F}.\hat{n}dS = 27(4x\sin^3\theta-\cos^3\theta)\,d\theta dx$$

$$\int_{S_5}\vec{F}.\hat{n}\,dS = 27\int_0^2\int_0^{\pi/2}(4x\sin^3\theta-\cos^3\theta)\,d\theta dx$$

$$\left[\int_0^{\pi/2}\sin^3\theta d\theta = \int_0^{\pi/2}\cos^3\theta d\theta = \frac{\Gamma 2\,\Gamma \tfrac{1}{2}}{2\Gamma \tfrac{5}{2}} = \frac{\Gamma 2\,\Gamma\tfrac{1}{2}}{2.\frac{3}{2}.\frac{1}{2}\Gamma\tfrac{1}{2}} = \frac{2}{3}\right]$$

$$= 18\int_0^2(4x-1)\,dx$$

$$= 18[2x^2 - x]_0^2$$
$$= 108$$

So, surface integral $\oint_S \vec{F}.\hat{n}dS$ is given as

$$\oint_S \vec{F}.\hat{n}dS = \int_{S_1} \vec{F}.\hat{n}dS + \int_{S_2} \vec{F}.\hat{n}dS + \int_{S_3} \vec{F}.\hat{n}dS + \int_{S_4} \vec{F}.\hat{n}dS + \int_{S_5} \vec{F}.\hat{n}dS$$
$$= 0 + 72 + 0 + 0 + 108$$
$$= 180$$

So, $$\oint \vec{F}.\hat{n}dS = \int \nabla.\vec{F}d\tau$$

Hence, Gauss divergence theorem is verifed

19. Evaluate by using Gauss divergence theorem

(i) $\oint_s (a^2x^2 + b^2y^2 + c^2z^2)^{1/2} dS$

(ii) $\oint_s (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2} dS$

over the ellipsoid $ax^2 + by^2 + cz^2 = 1$

Solution.

S is the ellipsoid belonging to family to level surface as shown in Figure 7.22.

$S : ax^2 + by^2 + cz^2 =$ constant

The outward drawn unit normal vector $\hat{n}$ to S is given by $\hat{n} = \dfrac{ax\hat{i} + by\hat{j} + cz\hat{k}}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$

(i) $\oint (a^2x^2 + b^2y^2 + c^2z^2)^{1/2} dS = \oint \vec{F}.\hat{n}dS$

Comparing the integrals

$$\vec{F} \cdot \hat{n} = (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}$$
$$\vec{F} \cdot \frac{(ax\hat{i} + by\hat{j} + cz\hat{k})}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}$$
$$\vec{F} \cdot (ax\hat{i} + by\hat{j} + cz\hat{k}) = a^2x^2 + b^2y^2 + c^2z^2$$

For using Gauss divergence theorem, $\vec{F}$ should continuous and should have continuous partial derivatives in region V enclosed by ellipsoid S. The surface $\vec{F}$ can be taken as

$$\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$$

So, $$\oint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2} dS = \oint_S (ax\hat{i} + by\hat{j} + c^2\hat{k}).\hat{n}dS$$
$$= \int_V \nabla.(ax\hat{i} + by\hat{j} + cz\hat{k})d\tau$$

According to Gauss Divergence theorem

$$\oint \vec{F}.\hat{n}dS = \int \nabla.\vec{F}d\tau$$
$$= (a+b+c)\int_V d\tau$$
$$= (a+b+c) \times \text{volume of ellipsoid}$$
$$= \frac{4\pi(a+b+c)}{3\sqrt{abc}}$$

(ii) $\oint(a^2x^2+b^2y^2+c^2z^2)^{-1/2}dS = \oint\vec{F}.\hat{n}dS$

Comparing the integral

$\vec{F}\cdot\hat{n} = (a^2x^2+b^2y^2+c^2z^2)^{-1/2}$

$$\Rightarrow \quad \vec{F}\cdot\frac{(ax\hat{i}+by\hat{j}+cz\hat{k})}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

$$\Rightarrow \quad \vec{F}\cdot(ax\hat{i}+by\hat{j}+cz\hat{k}) = 1$$

The function $\vec{F}$ can be taken as

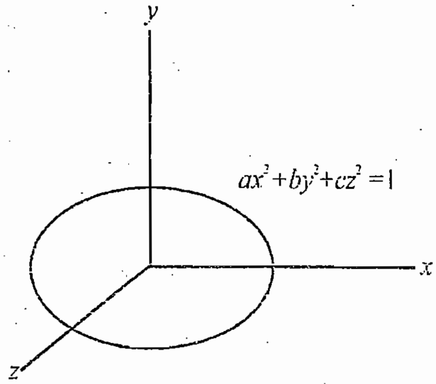


Figure 7.22

$$\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$\vec{F}.\hat{n} = (x\hat{i}+y\hat{j}+z\hat{k})(ax\hat{i}+by\hat{j}+cz\hat{k})$$

$$= ax^2+by^2+cz^2 = 1 \quad (\text{on } S,\ ax^2+by^2+cz^2=1)$$

$$\oint_S(ax^2+by^2+cz^2)dS = \oint(x\hat{i}+y\hat{j}+z\hat{k}).\hat{n}dS$$

$$= \int\nabla\cdot(x\hat{i}+y\hat{j}+z\hat{k})d\tau$$

$$= 3\int d\tau$$

$$= 3\times\text{volume of ellipsoid}$$

$$= \frac{4\pi}{\sqrt{abc}}$$

Note : While evaluating surface integration, we can incorporate the equation of surface.

20. If $\vec{F}=(x^2+y-4)\hat{i}+3xy\,\hat{j}+(2xz+z^2)\hat{k}$. Evaluate $\int_S(\nabla\times\vec{F}).\hat{n}dS$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above xy plane.

Solution.

The surface S is sphere $x^2+y^2+z^2=a^2$ above xy plane as shown is Figure 7.23.

So, S is a open surface. But, Gauss theorem applies only to surface integral on closed surface. Had the surface S been closed, the integral $\int_S\nabla\times\vec{F}.\hat{n}dS$ would have been zero because

$$\int_S\nabla\times\vec{F}.\hat{n}dS = \int\nabla.(\nabla\times\vec{F})d\tau = 0.$$

Since, divergence of curl $\vec{F}$ will be zero.

S is open surface. Here we will make use of the fact that $\int\nabla\times\vec{F}.\hat{n}dS$ over the closed surface will be zero.

Now, let us consider a closed surface Σ consisting of hemispherical part $S: x^2+y^2+z^2=a^2$ above xy plane and base of hemisphere $S': z=0$.

We have to find $\int_S\nabla\times\vec{F}.\hat{n}dS$

Now, $$\oint_\Sigma\nabla\times\vec{F}.\hat{n}dS = \int\nabla\cdot(\nabla\times\vec{F})d\tau = 0$$

$$\Rightarrow \quad \oint_\Sigma\nabla\times\vec{F}.\hat{n}dS = \int_S\nabla\times\vec{F}.\hat{n}dS+\int_{S'}\nabla\times\vec{F}.\hat{n}dS = 0$$

$$\int_S\nabla\times\vec{F}.\hat{n}dS = -\int_{S'}\nabla\times\vec{F}.\hat{n}dS$$

Now, it is easier to evaluate the surface integral over the plane surface ie. over the base of hemisphere $S' : z = 0$.

On S', $z=0$, $dS = dxdy$, $\hat{n} = -\hat{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$= -2z\hat{j} + (3y-1)\hat{k}$$

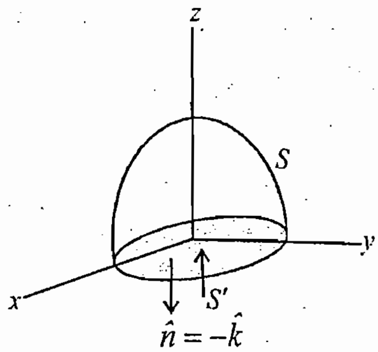


Figure 7.23

So, $\nabla\times\vec{F}.\hat{n} = (-2z\hat{j}+(3y-1)\hat{k})\cdot(-\hat{k}) = -(3y-1)$

So, $\int_S \nabla\times\vec{F}.\hat{n}dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}dS = \int\int(3y-1)dydx$

$$= 3\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y\,dydx - \int\int dydx$$

$$= 0 - \text{Area of base}$$

$$= -\pi a^2$$

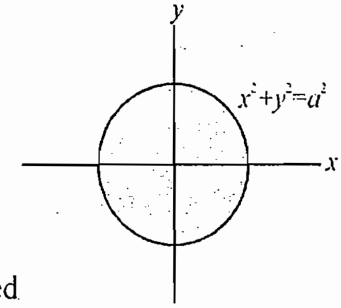


Figure 7.24

Note : In this problem, we have converted as integral over a curved surface to integral over a plane surface.

21. Evaluate $\int_S (\nabla\times\vec{F})\cdot\hat{n}dS$ where $\vec{F} = (x-z)\hat{i} + (x^3+yz)\hat{j} - 3xy^2\hat{k}$ and S is the surface of the cone $z = 2-\sqrt{x^2+y^2}$ above the xy plane.

Solution.

The general equation of cone with axis parallel to z axis and vertex at (α,β,γ) with semivertical angle θ is given by.

$$(z-\gamma)\tan^2\theta = (x-\alpha)^2 + (y-\beta)^2$$

The cone given by above equation is shown in Figure 7.25

$(z-\gamma)\tan\theta = +\sqrt{(x-\alpha)^2+(y-\beta)^2}$ denotes part of cone above the vertex (α,β,γ) as shown in Figure 7.26.

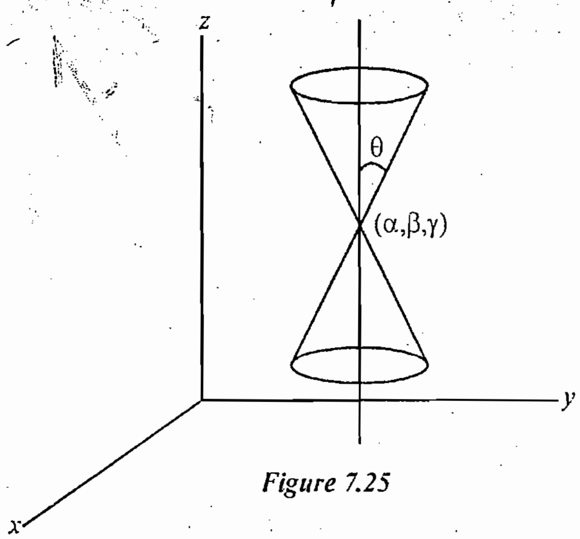


Figure 7.25

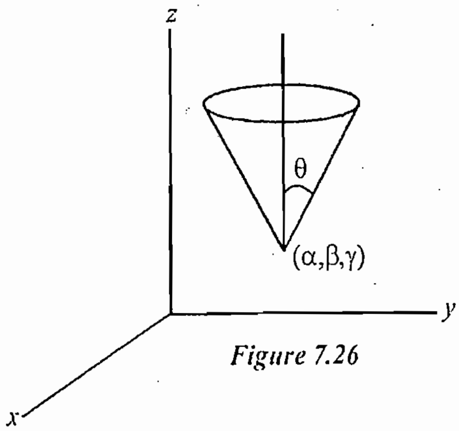


Figure 7.26

$(z-\gamma)\tan\theta = -\sqrt{(x-\alpha)^2+(y-\beta)^2}$ denotes the part of cone below the vertex (α,β,γ) as shown in Figure 7.27

Equation of cone given here is

$$z = 2-\sqrt{x^2+y^2}$$

$$(z-2) = -\sqrt{x^2+y^2}$$

Comparing this equation with standard equation of cone

$$(z-\gamma)\tan\theta = \sqrt{(x-\alpha)^2+(y-\beta)^2}$$

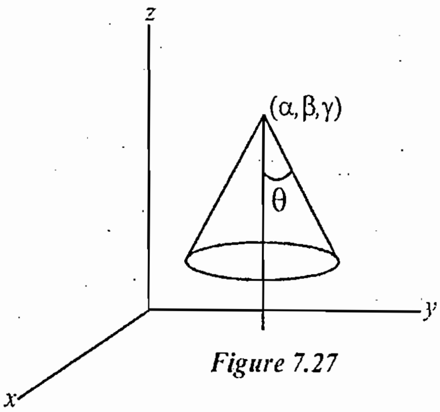


Figure 7.27

The vertex is (0,0,2) and semivertical angle is θ. It represents part of cone below the vertex. Here also, we will make use of the fact that the surface integral $\int \nabla\times\vec{F}.\hat{n}dS$ over the closed surface will be zero

as $$\oint \nabla\times\vec{F}.\hat{n}dS = \int \nabla.(\nabla\times\vec{F})d\tau = 0$$

Since $$\nabla.(\nabla\times\vec{F}) = 0$$

Let us consider a closed piecewise smooth surface Σ consisting of two surface.

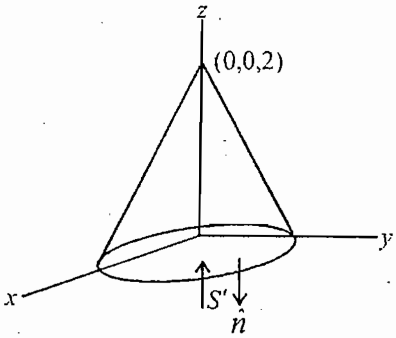


Figure 7.28

S: Part of cone $z=2-\sqrt{x^2+y^2}$ lying above xy plane, S' : base of cone, bounded by $x^2+y^2=4$, $z=0$.

The surface integral

$$\oint_\Sigma \nabla\times\vec{F}.\hat{n}dS = \int \nabla.(\nabla\times\vec{F})d\tau = 0$$

$$\Rightarrow \quad \oint_\Sigma \nabla\times\vec{F}.\hat{n}dS = \int_S \nabla\times\vec{F}.\hat{n}dS + \int_{S'} \nabla\times\vec{F}.\hat{n}dS = 0$$

$$\Rightarrow \quad \int_S \nabla\times\vec{F}.\hat{n}dS = -\int_{S'} \nabla\times\vec{F}.\hat{n}dS$$

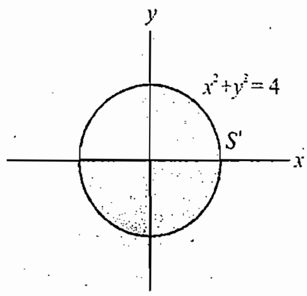


Figure 7.29

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x-z & x^3+yz & -3xy^2 \end{vmatrix}$$

$$= (-6xy-y)\hat{i}+(-1+3y^2)\hat{j}+(3x^2)\hat{k}$$

$$(\nabla\times\vec{F}).\hat{n} = -3x^2 \qquad (\hat{n}=-\hat{k})$$

So, $$\int_S \nabla\times\vec{F}.\hat{n}dS = -\int_{S'} \nabla\times\vec{F}.\hat{n}dS$$

$$= 3\int\int x^2 dxdy$$

$$= 3\int_0^{2\pi}\int_0^2 r^2\cos^2\theta r\,dr\,d\theta$$

$$= 3\int_0^{2\pi} \left.\frac{r^4}{4}\right|_0^2 \cos^2\theta d\theta$$

$$= 12\int_0^{2\pi}\cos^2\theta d\theta$$

$$= 12\pi$$

22. If $\vec{F} = y\hat{i}+(x-2xz)\hat{j}-xy\hat{k}$, Evaluate $\int_S \nabla\times\vec{F}.\hat{n}dS$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy plane.

Solution.

Similar to previous problem, here also use will make use of the fact that the surface integral $\int \nabla\times\vec{F}\cdot\hat{n}\,dS$ evaluated over a closed surface is equal to zero because the divergence of function $\nabla\times\vec{F}$ is always zero.

So, applying Gauss divergence theorem

$\int \nabla\times\vec{F}\cdot\hat{n}dS$ evaluated over a closed surface is zero. Consider a closed piecewise smooth surface Σ consisting of two surfaces.

(i) S : spherical part $x^2+y^2+z^2=a^2$ above xy plane

(ii) S' : base of sphere $x^2+y^2+z^2=a^2$, bounded by circle $x^2+y^2=a^2$ in xy plane

Applying Gauss Divergence theorem

$$\oint_{\Sigma} \nabla\times\vec{F}.\hat{n}dS = \int_V \nabla.(\nabla\times\vec{F})d\tau = 0$$

$$\Rightarrow \quad \oint_{\Sigma} \nabla\times\vec{F}.\hat{n}dS = \int_S \nabla\times\vec{F}.\hat{n}dS + \int_{S'} \nabla\times\vec{F}.\hat{n}dS = 0$$

$$\Rightarrow \quad \int_S \nabla\times\vec{F}.\hat{n}dS = -\int_{S'} \nabla\times\vec{F}.\hat{n}dS$$

On S', $dS = dxdy$, $z=0$, $\hat{n}=-\hat{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix}$$

$$= x\hat{i}+y\hat{j}-2z\hat{k}$$

$$\nabla\times\vec{F}.\hat{n} = 2z = 0$$

$$\int_S \nabla\times\vec{F}.\hat{n}dS = -\int_{S'} \nabla\times\vec{F}.\hat{n}dS = 0$$

So, $\displaystyle\int_S \nabla\times\vec{F}.\hat{n}dS = 0$

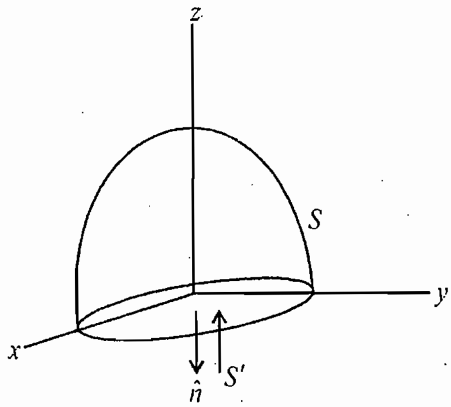


Figure 7.30

23. Evaluate $\int_S (\nabla\times\vec{F}).\hat{n}dS$ where $\vec{F}=(x^2+y-4)\hat{i}+3xy\hat{j}+(2xz+z^2)\hat{k}$ and S is the surface of paraboloid with axis parallel to z axis $z=9-(x^2+y^2)$.

Solution.

The standard equation of paraboloid is given by

$$(z-\gamma)=a\left[(x-\alpha)^2+(y-\beta)^2\right]$$

where (α,β,γ) is the vertex of paraboloid

Comparing given equation of paraboloid $z=9-(x^2+y^2)$ (Figure 7.31) with standard equation. The vertex is (0,0,9). Here also, we will make use of fact that the integral $\int \nabla\times\vec{F}.\hat{n}dS$ is equal to zero for a closed surface as shown in Figure 7.33.

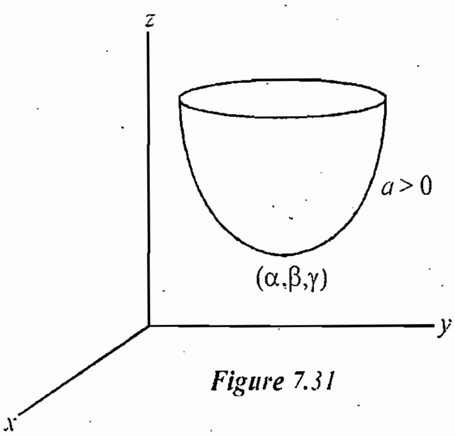


Figure 7.31

Consider a closed piecewise smooth surface Σ consisting of paraboloid S and base of paraboloid $x^2+y^2=9$ as shown in Figure 7.34.

Using Gauss Divergence theorem

$$\oint_{\Sigma} \nabla\times\vec{F}.\hat{n}dS = \int_V \nabla.(\nabla\times\vec{F})d\tau = 0$$

$$\Rightarrow \quad \oint_{\Sigma} \nabla\times\vec{F}.\hat{n}dS = \int_S \nabla\times\vec{F}.\hat{n}dS + \int_{S'} \nabla\times\vec{F}.\hat{n}dS = 0$$

So, $$\int_S \nabla\times\vec{F}.\hat{n}dS = -\int_{S'} \nabla\times\vec{F}.\hat{n}dS$$

On S', $\hat{n} = -\hat{k}$, $dS = dxdy$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$= -2z\hat{j} + (3y-1)\hat{k}$$

$$\nabla\times\vec{F}.\hat{n} = -(3y-1)$$

$$\int_S \nabla\times\vec{F}.\hat{n}dS = -\int_{S'} \nabla\times\vec{F}.\hat{n}dS$$

$$= \iint (3y-1)\,dydx$$

$$= 3\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} ydydx - \iint dydx$$

$= -$Area of base of paraboloid

$$= -9\pi$$

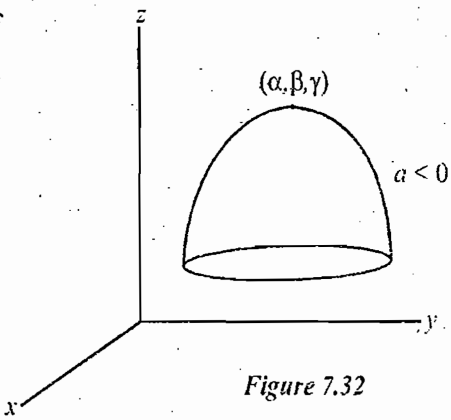


Figure 7.32

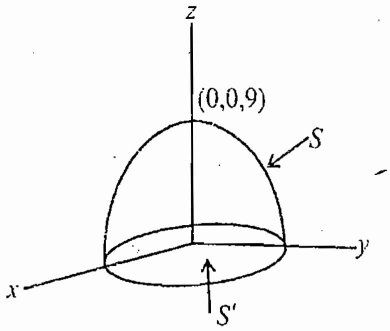


Figure 7.33

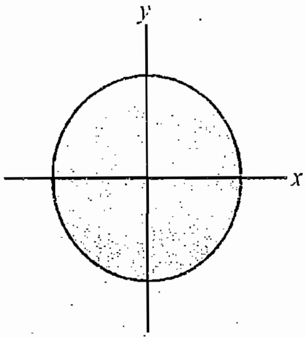


Figure 7.34

24. Let $\phi(x,y,z) = e^x \sin y$. Evaluate the surface integral $\iint_S \frac{\partial\phi}{\partial n} d\sigma$, where S is the surface of the cube $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$ and $\frac{\partial\phi}{\partial n}$ is the directional derivative of ϕ in the direction of the unit outward normal to S. verify the divergence theorem.

Solution.

$$\frac{\partial\phi}{\partial n} = \nabla\phi\cdot\hat{n}$$

$$\nabla\phi = e^x \sin y\hat{i} + e^x \cos y\hat{j}$$

S is a piecewise smooth surface as shown in Figure 7.35 consisting of following surface

$x=0,\quad x=1$

$y=0,\quad y=1$

$z=0,\quad z=1$

On $x=0$, $\hat{n} = -\hat{i}$, $dS = dydz$

$$\iint \frac{\partial\phi}{\partial n} d\sigma = -\int_0^1\int_0^1 \sin y dydz$$

$$= (\cos 1 - 1)$$

On $x = 1$; $\hat{n} = \hat{i}$; $dS = dydz$

$$\iint \frac{\partial \phi}{\partial n} \cdot d\sigma = \int_0^1\int_0^1 e \sin y \, dydz$$
$$= -e(\cos 1 - 1)$$

On $y = 0$, $\hat{n} = -\hat{j}$, $dS = dxdz$

$$\iint \frac{\partial \phi}{\partial n} d\sigma = -\iint e^x dxdz$$
$$= -\int_0^1\int_0^1 e^x dxdz$$
$$= -(e-1)$$

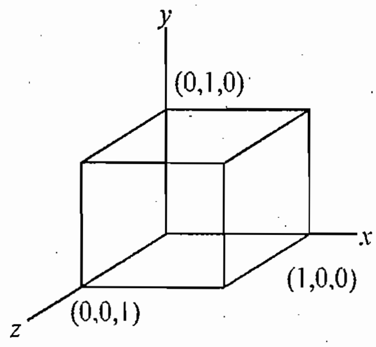


Figure 7.35

On $y = 1$, $\hat{n} = \hat{j}$, $dS = dxdz$

$$\iint \frac{\partial \phi}{\partial n} d\sigma = \int_0^1\int_0^1 e^x \cos 1 dxdz$$
$$= \cos 1(e-1)$$

On $z = 0$, $\hat{n} = -\hat{k}$, $dS = dxdy$

$$\iint \frac{\partial \phi}{\partial n} d\sigma = 0$$

On $z = 1$, $dS = dxdz$ $z = 1, \hat{n} = \hat{k}, dS = dxdy$

$$\iint \frac{\partial \phi}{\partial n} d\sigma = 0$$

So, $$\oint \frac{\partial \phi}{\partial n} \partial\sigma = (\cos 1 - 1) - e(\cos 1 - 1) - (e-1) + \cos 1(e-1)$$
$$= 0$$

Using Divergence Theorem

$$\oint_S \nabla \phi \cdot \hat{n} dS = \int_V \nabla^2 \phi d\tau = 0$$

Since, $$\nabla^2 \phi = \nabla \cdot (e^x \sin y \hat{i} + e^x \cos y \hat{j})$$
$$= e^x \sin y - e^x \sin y = 0$$

Hence, Gauss Divergence theorem is verified

25. Let S be the surface $\{(x, y, z) \in \boldsymbol{R}^3 : x^2 + y^2 + 2z = 2, z \geq 0\}$, and let $\hat{n}$ be the outward unit normal to S. If $\vec{F} = y\hat{i} + xz\hat{j} + (x^2 + y^2)\hat{k}$, then evaluate the integral $\int_S \vec{F} \cdot \hat{n} dS$.

Solution

$S : x^2 + y^2 = -2(z-1)$

is a paraboloid with vertex at (0, 0, 1) as shown is Figure 7.36

$$\vec{F} = y\hat{i} + xz\hat{j} + (x^2 + y^2)\hat{k}$$
$$\nabla \cdot \vec{F} = 0$$

Consider a closed surface S which consists of two piecewise smooth surface S and S', where S' is base of Paraboloid and S is paraboloid

$$\oint_\Sigma \vec{F} \cdot \hat{n} dS = \int \nabla \cdot \vec{F} d\tau = 0$$
$$\oint_\Sigma \vec{F} \cdot \hat{n} dS = \int_S \vec{F} \cdot \hat{n} dS + \int_{S'} \vec{F} \cdot \hat{n} dS = 0$$
$$\int_S \vec{F} \cdot \hat{n} dS = -\int_{S'} \vec{F} \cdot \hat{n} dS$$

For S', $\hat{n} = -\hat{k}$ $\quad dS = dxdy$

So, $$\int_S \vec{F}\cdot\hat{n}\,dS = -\int_{S'} \vec{F}\cdot\hat{n}dS$$
$$= -\iint (y\hat{i} + xz\hat{j} + (x^2+y^2)\hat{k}).(-\hat{k})dxdy$$
$$= \iint (x^2+y^2)\ dxdy$$
$$= \int_0^{2\pi}\int_0^{\sqrt{2}} r^2 r\,d\theta\,dr$$
$$= \int_0^{2\pi} \left|\frac{r^4}{4}\right|_0^{\sqrt{2}} d\theta = 2\pi$$

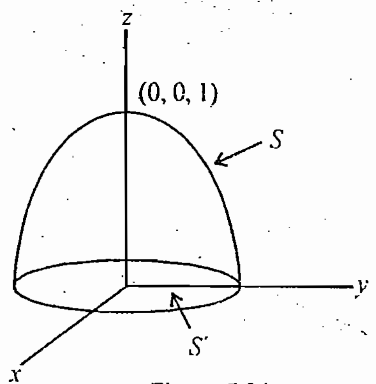


Figure 7.36

26. Let D be the region bounded by the concentric spheres $S_1 : x^2+y^2+z^2 = a^2$ and $S_2 : x^2+y^2+z^2 = b^2$, where $a < b$. Let $\hat{n}$ be the unit normal to S_1 directed away from the origin. If $\nabla^2\varphi = 0$ in D and $\varphi = 0$ on S_2, then show that $\int_D |\nabla\phi|^2\, dV + \int_{S_1} \phi(\nabla\phi).\hat{n}dS = 0$.

Solution.

Let us consider a surface Σ consisting of S and S' enclosing a volume D as shown is Figure 7.37. According to Gauss Divergence theorem

$$\oint_\Sigma \phi\nabla\phi.\hat{n}dS = \int_D \nabla.(\phi\nabla\phi)dV$$
$$= \int_D \nabla\phi.\nabla\phi dV + \int_D \phi\nabla^2\phi dV$$
$$= \int_D |\nabla\phi|^2 dV \text{ (as } \nabla^2\phi = 0 \text{ in } D)$$

Now, $$\oint_\Sigma \phi\nabla\phi\cdot\hat{n}dS = \int_{S_1} \phi\nabla\phi\cdot\hat{n}dS + \int_{S_2} \phi\nabla\phi\cdot\hat{n}dS$$
$$= \int_{S_1} \phi\cdot\nabla\phi\cdot\hat{n}dS + 0 \qquad (\text{as } \phi = 0 \text{ on } S_2)$$

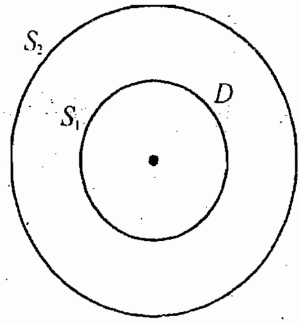


Figure 7.37

Here $\hat{n}$ is outward drawn normal i.e. pointing towards origin

So, $$\int_{S_1} \phi\cdot\nabla\phi\cdot\hat{n}dS = -\int_{S_1} \phi\cdot\nabla\phi\cdot\hat{n}'dS$$

where $\hat{n}' = -\hat{n}$ is unit normal to S_1 directed away from the origin

So, $$\int_D |\nabla\phi|^2 dV = -\int_{S_1} \phi\cdot\nabla\phi\cdot\hat{n}'dS$$
$$\int_D |\nabla\phi|^2 dV + \int_{S_1} \phi(\nabla\phi)\cdot\hat{n}'dS = 0$$

27. Using Gauss's divergence theorem, evaluate the integral $\int_S \vec{F}\cdot\hat{n}dS$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + 4yz\hat{k}$, S is the surface of the solid bounded by the sphere $x^2 + y^2 + z^2 = 10$ and the paraboloid $x^2+y^2 = z-2$, and $\hat{n}$ is the outward unit normal vector to S.

Solution.

$$\vec{F} = 4xz\hat{i} - y^2\hat{j} + 4yz\hat{k}$$
$$\nabla\cdot\vec{F} = 4z + 2y$$

Using Gauss divergence theorem

$$\oint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$$

$$= \iint\int_{x^2+y^2+2}^{\sqrt{10-x^2-y^2}} (4z+2y)dzdydx$$

$$(d\tau = dxdydz)$$

$$= 2\iint\left[z^2+yz\right]_{x^2+y^2+2}^{\sqrt{10-x^2-y^2}} dxdy$$

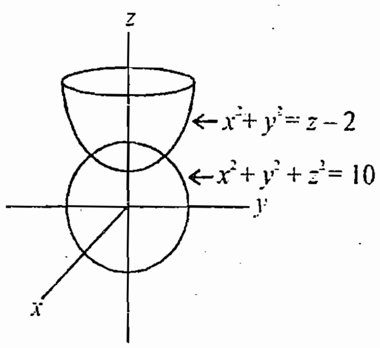


Figure 7.38

$$= 2\int\int(6-5(x^2+y^2)-(x^2+y^2)^2+y\left(\sqrt{10-x^2-y^2}-x^2-y^2-2\right)dxdy$$

Surfaces bounding the volume are $x^2+y^2+z^2=10$ & $x^2+y^2=z-2$ as shown in Figure 7.38.

So, curve of intersection o! › surfaces is given as

$z^2+z-2=10 \Rightarrow z=3$

$$\left.\begin{aligned} x^2+y^2&=1 \\ z&=3 \end{aligned}\right\} \text{Curve of intersection}$$

Putting $x=r\cos\theta, y=r\sin\theta, dxdy = rd\theta dr$
(r is the region of integration of double integration)

$$\oint_S \vec{F}\cdot\hat{n}dS = 2\int_0^1\int_0^{2\pi}\left[(6-5r^2-r^4)+r\sin\theta\left(\sqrt{10-r^2}-(r^2+2)\right)\right]rd\theta dr$$

$$= 2\int_0^1\int_0^{2\pi}(6-5r^2-r^4)rd\theta dr + 2\int_0^1\int_0^{2\pi} r^2\left(\sqrt{10-r^2}-r^2-2\right)\sin\theta d\theta dr$$

Now, $\int_0^{2\pi}\sin\theta d\theta = 0$ So, integral of second term

$$2\int_0^1\int_0^{2\pi} r^2\left(\sqrt{10-r^2}-r^2-2\right)\sin\theta d\theta dr = 0$$

$$\oint \vec{F}\cdot\hat{n}dS = 4\pi\int_0^1\left[6r-5r^3-r^5\right]dr$$

$$= 4\pi\left[3r^2-\frac{5r^4}{4}-\frac{r^6}{6}\right]_0^1 = \frac{19}{3}\pi$$

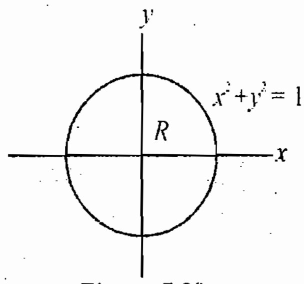


Figure 7.39

28. Let W be the region bounded by the planes $x=0, y=0, y=3, z=0$ and $x+2z=6$. Let S be the boundary of this region. Using Gauss divergence theorem, evaluate $\int_S \vec{F}\cdot\hat{n}dS$, where $\vec{F} = 2xy\hat{i}+yz^2\hat{j}+xz\hat{k}$ and $\hat{n}$ is the outward unit normal vector to S.

Solution.

Using Gauss Divergence theorem

$$\oint \vec{F}\cdot\hat{n}dS = \int\nabla\cdot Fd\tau$$

$$= \iiint(2y+z^2+x)dxdydz$$

$$= \iint\int_0^3(x+2y+z^2)dydxdz$$

$$= \iint xy+y^2+z^2y\Big|_0^3 dxdz$$

$$= \int_0^6\int_0^{\frac{6-x}{2}}(3x+3z^2+9)dzdx$$

$$= \int_0^6 3xz + 9z + z^3 \Big|_0^{\frac{6-x}{2}} dx$$

$$= \frac{1}{8}\int_0^6 (-x^3 + 6x^2 - 72x + 512)dx$$

$$= \frac{1}{8}\left[-\frac{x^4}{4} + 2x^3 - 36x^2 + 512x\right]_0^6$$

$$= 235.5$$

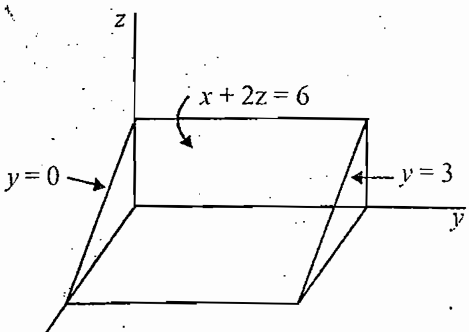


Figure 7.40

29. If $\vec{F} = (x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xz + z^2)\hat{k}$, then evaluate the surface integral $\int_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$, where S is the surface of the cone $z = 1 - \sqrt{x^2 + y^2}$ lying above the xy-plane and $\hat{n}$ is the unit normal to S making an acute angle with $\hat{k}$.

Solution.

Consider a closed surface Σ consisting of S & S' as shown in Figure 7.41. Where S is conical surface & S' is its base

$$\oint_\Sigma \nabla \times \vec{F} \cdot \hat{n} dS = \int_V \nabla.(\nabla \times \vec{F}) d\tau = 0$$

$$\Rightarrow \quad \int_S \nabla \times \vec{F} \cdot \hat{n} dS + \int_{S'} \nabla \times \vec{F} \cdot \hat{n} dS = 0$$

$$\int_S \nabla \times \vec{F} \cdot \hat{n} dS = -\int_{S'} \nabla \times \vec{F} \cdot \hat{n} dS$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y - 4 & 3x & 2xz + z^2 \end{vmatrix}$$

$$= -2z\,\hat{j} + 2\,\hat{k}$$

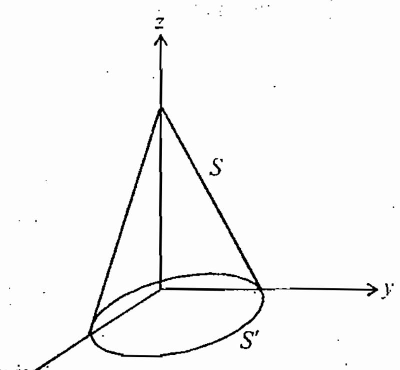


Figure 7.41

For S', $\hat{n} = -\hat{k}$, $dS = dxdy$

So, $\int_S \nabla \times \vec{F} \cdot \hat{n} dS = -\int \nabla \times \vec{F} \cdot \hat{n} dS$

$$= \iint 2\, dxdy$$

$$= 2\pi$$

30. For the vector field

$\vec{V} = xz^2\hat{i} - yz^2\hat{j} + z(x^2 - y^2)\hat{k}$

(a) Calculate the volume integral of the divergence of $\vec{V}$ over the region defined by $-a \le x \le a$, $-b \le y \le b$ and $0 \le z \le c$.

(b) Calculate the flux of $\vec{V}$ out of the region through the surface at $z = c$. Hence deduce the net flux through the rest of the boundary of the region.

Solution.

$$\vec{V} = xz^2\hat{i} - yz^2\hat{j} + z(x^2 - y^2)\hat{k}$$

$$\text{div}\,\vec{V} = z^2 - z^2 + x^2 - y^2$$

$$= (x^2 - y^2)$$

Volume integral of divergence of $\vec{V}$ over the region defined by $-a \le y \le a$, $-b \le y \le b$ and $0 \le z \le c$.

$$\iiint \nabla\cdot\vec{V}\,dx\,dy\,dz = \int_{-a}^{a}\int_{-b}^{b}\int_{0}^{c}(x^2-y^2)dxdydz$$

$$= c\int_{-a}^{a}\int_{-b}^{b}(x^2-y^2)dxdy$$

$$= c\int_{-a}^{a}\left|x^2y-\frac{y^3}{3}\right|_{-b}^{b} dx$$

$$= 2c\int_{-a}^{a}\left(bx^2-\frac{b^3}{3}\right)dx$$

$$= 2c\left|\frac{bx^3}{3}-\frac{b^3x}{3}\right|_{-a}^{a}$$

$$= \frac{4abc}{3}(a^2-b^2)$$

(b) For $z=c,\ \hat{n}=\hat{k}$

$$\vec{V}\cdot\hat{n} = z(x^2-y^2) = c(x^2-y^2) \text{ for } z=c$$
$$dS = dxdy$$

Flux across $z=c$,

$$\int\vec{V}\cdot\hat{n}dS = c\int_{-a}^{a}\int_{-b}^{b}(x^2-y^2)dxdy$$

$$= c\int_{-a}^{a}\left[x^2y-\frac{y^3}{3}\right]_{-b}^{b} dx$$

$$= 2c\int_{-a}^{a}\left(x^2b-\frac{b^3}{3}\right)dx = 2c\left[\frac{x^3b}{3}-\frac{b^3x}{3}\right]_{-a}^{a}$$

$$= \frac{4abc}{3}(a^2-b^2)$$

Flux across the closed surface

$$\oint\vec{V}\cdot\hat{n}\,dS = \int_{\tau}\nabla\cdot\vec{V}d\tau = \frac{4abc}{3}(a^2-b^2)$$

This is equal to flux through the surface at $z=c$

So, flux through rest of the boundary of the region $=0$

31. Using Divergence theorem, evaluate $\int_S \vec{F}.\hat{n}dS$, where $\vec{F}=4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$ and S is the surface bounded by the region $x^2+y^2=4,\ z=0,\ z=3$.

Solution.

From Divergence Theorem

$$\int_S\vec{F}.\hat{n}dS = \int_V\nabla\cdot\vec{F}d\tau$$

$$\nabla\cdot\vec{F} = (4-4y+2z)$$

$$\int_V\nabla\cdot\vec{F}d\tau = \iint\int_{z=0}^{3}(4-4y+2z)dxdydz$$

$$= \iint\left|4z-4yz+z^2\right|_0^3 dxdy$$

$$= \iint (21-12y)\, dxdy$$

$$= 21\iint dxdy - 12\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} ydxdy$$

$$= 21\iint dxdy - 0 \quad \left[\because \int_{-a}^{a} f(x)dx = 0 \text{ if } f(x) \text{ is odd function}\right]$$

$= 21 \times$ area of circle of radius $2 = 84\pi$

32. Let S be the boundary of the region consisting of the parabolic cylinder $z = 1-x^2$ and the planes $y=0, y=2$ and $z=0$. Evaluate the integral $\oint_S \vec{F}.\hat{n}\, dS$, where $\vec{F} = xy\hat{i} + (y^2 + e^{xz^2})\hat{j} + \sin(xy)\hat{k}$ and $\hat{n}$ is the outward drawn unit normal to S.

Solution.

The surface S is shown Figure 7.42

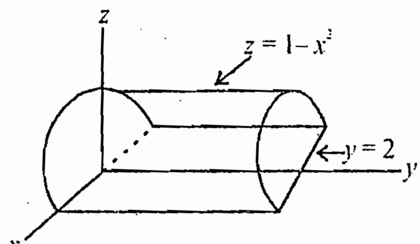


Figure 7.42

$$\oint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$$

$$= \iint\int_0^2 3y\, dx\, dz\, dy = 3\iint \frac{y^2}{2}\Big|_0^2 dx\, dz$$

$$= 6\int_{-1}^{1}\int_0^{1-x^2} dx\, dz = 6\int_{-1}^{1}(1-x^2)\, dx$$

$$= 6\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 12\times\frac{2}{3} = 8$$

33. If $\vec{V} = x^2z\hat{i} + y\hat{j} - xz^2\hat{k}$ and S is the surface of the closed cylinder $x^2+y^2 = 16$, $z=0$ and $z=4$, evaluate the integral $\iint_S \vec{V}.\hat{n}dS$.

Solution.

$$\oint \vec{V}\cdot\hat{n}\, dS = \int \nabla\cdot\vec{V}\, d\tau \qquad \text{(Gauss Divergence Theorem)}$$

$$= \int (2xz + 1 - 2xz)d\tau$$

$=$ volume of cylinder

$= 64\pi$

7.2 SOLID ANGLE

Just like a curve ds substend an angle $d\theta$ at a point

$$d\theta = \frac{ds}{r}$$

The surface dS subtend solid angle at the given point O.

$$d\Omega = \frac{\vec{r}.\hat{n}dS}{r^3}$$

$$= \frac{\hat{e}_r.\hat{n}dS}{r^2} \qquad \left(\hat{e}_r = \frac{\vec{r}}{r}\right)$$

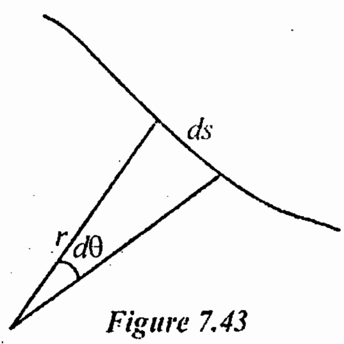


Figure 7.43

$\hat{e}_r$ is the unit vector in direction of increasing r.

$\hat{n}$ is an outward drawn unit normal vector to S.

Theorem. The solid angle subtended by any arbitrary surface S bounded by a closed curve C is same ie. This solid angle substended by any two surfaces bounded by same closed curve C is equal.

Proof: Let S is a open surface bounded by closed curve C. Let us take a plane surface S' bounded C. S and S' together enclose a region R as show in Figure 7.45.

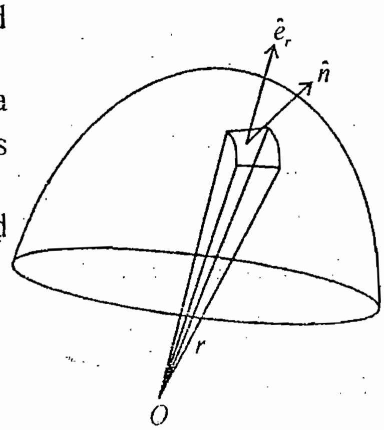


Figure 7.44

Consider a closed piecewise smooth surface Σ consisting of S and S', then by applying Gauss Divergence theorem

$$\oint_{\Sigma}\frac{\vec{r}}{r^3}.\hat{n}dS = \int\nabla.\left(\frac{\vec{r}}{r^3}\right)d\tau = 0$$

$$\Rightarrow \quad \int_S\frac{\vec{r}}{r^3}.\hat{n}dS + \int_{S'}\frac{\vec{r}}{r^3}.\hat{n}dS = 0$$

where $\hat{n}$ is an outward drawn normal to Σ

$$\Rightarrow \quad \int_S\frac{\vec{r}}{r^3}.\hat{n}dS = -\int_{S'}\frac{r}{r^3}.\hat{n}dS$$

$$= \int_{S'}\frac{\vec{r}}{r^3}.\hat{n}'dS \qquad (\hat{n}' = -n)$$

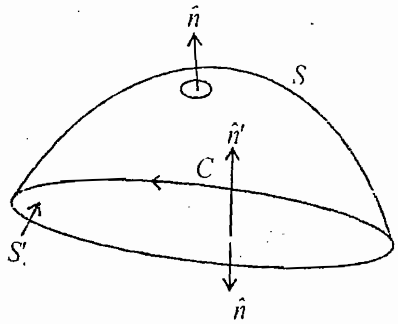


Figure 7.45

So, Solid angle subtended by S = Solid angle subtended by S'.

Hence, the solid angle subtended at O by any arbitrary surface S bounded by closed curve C is equal to the solid angle subtended at O by a plane surface S' bounded by the same curve C provided O lies outside the region enclosed by arbitrary surface S and plane surface S'.

Theorem : The solid angle subtended by an arbitrary closed surface S at point O is equal to 4π if O lies inside the the region V enclosed by closed surface S and zero if O lies outside S.

$$\oint_S\frac{\vec{r}}{r^3}.\hat{n}dS = 4\pi \qquad \text{if } O \text{ lies inside } S$$

$$= 0 \qquad \text{if } O \text{ lies outside } S$$

Proof: Case 1 : If O lies outside S. The vector function $\vec{F} = \frac{\vec{r}}{r^3}$ is continuous and has continuous partial derivatives at every point P lying in the region V enclosed by S.

So, Gauss divergence theorem can be applied. We also know that $\nabla.\left(\frac{\vec{r}}{r^3}\right) = 0$

So, the solid angle subtended by S at point O.

$$\oint_S\frac{\vec{r}}{r^3}.\hat{n}dS = \int_V\nabla.\left(\frac{\vec{r}}{r^3}\right)d\tau = 0$$

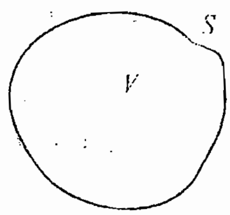


Figure 7.46

Case II : If O lies inside S.

The vector function $\vec{F} = \frac{\vec{r}}{r^3}$ is discontinuous at point O.

So, Gauss Divergence theorem cannot be applied to evaluate $\oint_S\frac{\vec{r}}{r^3}.\hat{n}dS$.

Let us enclose the point O by a sphere S' of very small radius $\in$.

Consider a closed surface Σ consisting two surface S & S' enclosing a region V', the function $\vec{F}=\frac{\vec{r}}{r^3}$ is continuous and has continuous partical derivaties at every point in V.

So, $$\oint_{\Sigma}\frac{\vec{r}}{r^3}.\hat{n}dS = \int \nabla\cdot\left(\frac{\vec{r}}{r^3}\right)d\tau = 0$$

$$\Rightarrow \quad \int_S \frac{\vec{r}}{r^3}.\hat{n}dS + \int_{S'}\frac{\vec{r}}{r^3}.\hat{n}dS = 0$$

So, $$\int_S \frac{\vec{r}}{r^3}.\hat{n}dS = -\int_{S'}\frac{\vec{r}}{r^3}.\hat{n}dS$$

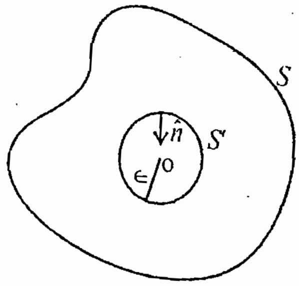


Figure 7.47

On S', $r=\in$ $\quad \hat{n} = -\frac{\vec{r}}{\in}$

So, $$\int \frac{\vec{r}}{r^3}.\hat{n}dS = -\int_{S'}\frac{\vec{r}}{r^3}.\hat{n}dS$$

$$= -\int_{S'}\frac{\vec{r}}{\in^3}.\left(-\frac{\vec{r}}{\in}\right).dS = \frac{1}{\in^4}\int_{S'}\vec{r}.\vec{r}dS$$

$$= \frac{1}{\in^4}.\int_{S'} r^2 dS = \frac{1}{\in^4}.\int_{S'}\in^2 dS$$

$$= \frac{1}{\in^2}\int dS = \frac{1}{\in^2}.4\pi\in^2$$

$$= 4\pi$$

Note : The concept of solid angle is very useful in evaluating flux across a surface of those vector fields $\vec{F}$ which follow inverse square law.

For example : The electrostatic field at any point P due to a discrete charge q is given as

$$\vec{E} = \frac{q}{4\pi\in_0 r^2}\hat{e}_r$$

where $\hat{e}_r$ is a unit vector is direction of increasing r, $\hat{e}_r = \frac{\vec{r}}{r}$

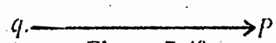


Figure 7.48

$$\vec{E} = \frac{q}{4\pi\in_0}\left(\frac{\vec{r}}{r^3}\right)$$

Gauss Law of Electrostatics : The flux of electric field $\vec{E}$ due to a point charge a across the surface S.

$$\oint \vec{E}.\hat{n}dS = \frac{q}{\in_0} \text{ If } q \text{ lies inside } S$$
$$= 0 \text{ If } q \text{ lies outside S.}$$

Proof : If q lies inside S

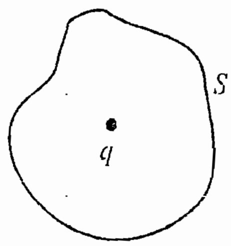


Figure 7.49

$$\oint \vec{E}.\hat{n}dS = \oint \frac{q}{4\pi\in_0}\frac{\vec{r}}{r^3}.\hat{n}dS$$

$$= \frac{q}{4\pi\in_0}\oint\frac{\vec{r}}{r^3}.\hat{n}dS$$

$$= \frac{q}{4\pi\in_0}\times \text{Solid angle subtended by } S \text{ at position of } q.$$

$$= \frac{q}{4\pi\in_0}\cdot 4\pi \qquad \text{(Using the results obtained is previous theorem)}$$

$$= \frac{q}{\in_0}$$

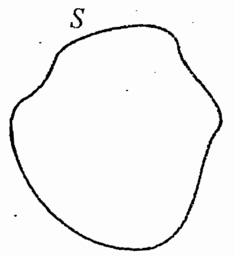


Figure 7.50

If q lies outside S.

$$\oint \vec{E}.\hat{n}dS = \oint \frac{q}{4\pi \in_0} \frac{\vec{r}}{r^3}.\hat{n}dS$$

$$= \frac{q}{4\pi \in_0} \oint \frac{\vec{r}}{r^3}.\hat{n}dS$$

$$= \frac{q}{4\pi \in_0} \times \text{Solid angle subtended by } S \text{ at the position of } q$$

$$= 0$$

34. Evaluate the solid angle subtended by any arbitrary open surface bounded by circle C at any point O.

Solution.

The solid angle subtended by S at O is same as solid angle subtended by a plane surface S'. Let us evaluate the solid angle subtended at point O by surface S'.

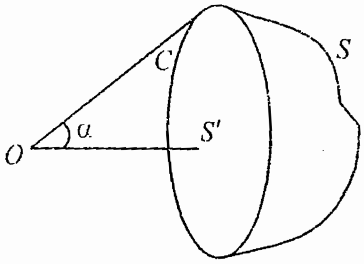


Figure 7.51

Consider a ring of radius x and thickness dx, the surface element

$$dS = 2\pi x dx$$

Solid angle subtended by surface elements dS at O.

$$d\Omega = \frac{\vec{r}.\hat{n}dS}{r^3} = \frac{\hat{e}_r.\hat{n}}{r^2}dS$$

$$= \frac{2.\pi x \cos\theta}{(x^2+d^2)}dx$$

$$= \frac{2\pi dx}{(x^2+d^2)^{3/2}}dx$$

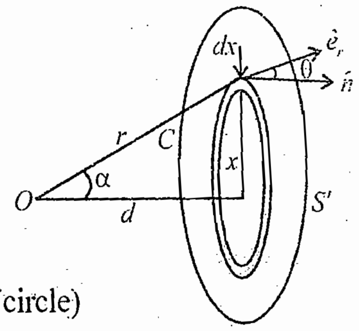


Figure 7.52

So, Solid angle subtended by S' at point O.

$$\Omega = \int d\Omega$$

$$= 2\pi d\int_0^R \frac{x}{(x^2+d^2)^{3/2}}dx \qquad (R \text{ is the radius of circle})$$

Let $\quad x^2 + d^2 = t^2$

$\Rightarrow \quad xdx = tdt$

$$\Omega = 2\pi d\int \frac{tdt}{t^3}$$

$$= 2\pi d\left[-\frac{1}{t}\right] = 2\pi d\left[-\frac{1}{\sqrt{x^2+d^2}}\right]_0^R$$

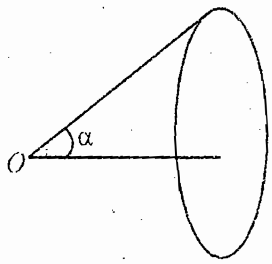


Figure 7.53

$$= 2\pi d\left[\frac{1}{d} - \frac{1}{\sqrt{R^2+d^2}}\right]$$

$$= 2\pi\left[1 - \frac{d}{\sqrt{R^2+d^2}}\right] = 2\pi(1-\cos\alpha)$$

So, the solid angle substended by a disc of radius R at any point O lying on a perpendicular axis passing through its centre is given as

$$\Omega = 2\pi(1-\cos\alpha)$$

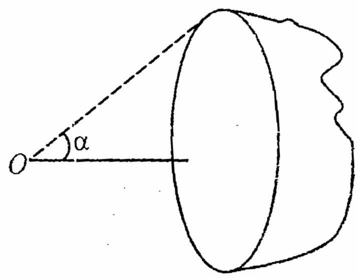


Figure 7.54

Since, the solid angle is same for any surface bounded by C. So, the solid angle subtended by any arbitrary surface S bounded by a circle C is equal to

$$\Omega = 2\pi(1-\cos\alpha)$$

35. Using result obtained in pervious problem, find the flux of electrostatics field across the disc of radius R due to point charge q placed at distance d from its centre.

Solution.

The flux of electrostatic field $\vec{E}$ across the disc S is given as

$$\phi = \int \vec{E}.\hat{n}dS$$

$$= \int \frac{q}{4\pi\in_0}\frac{\vec{r}}{r^3}.\hat{n}dS \text{ where } \hat{n} \text{ is unit normal vector to } S.$$

$$= \frac{q}{4\pi\in_0}\int\frac{\vec{r}}{r^3}.\hat{n}dS$$

$$= \frac{q}{4\pi\in_0} \times \text{Solid angle subtended by } S \text{ at the postition of } q.$$

$$= \frac{q}{4\pi\in_0}.2\pi(1-\cos\alpha)$$

$$= \frac{q}{2\in_0}\left(1-\frac{d}{\sqrt{R^2+d^2}}\right)$$

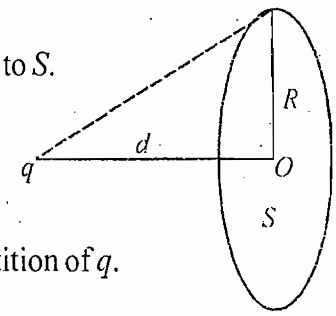


Figure 7.55

36. Evaluate the solid angle subtended by a part of sphere at centre O as shown in fig.

Solution.

The bounding curve of S is a circle.

So, solid angle subtended by S at centre O is same as solid angle subtended by a plane surface enclosed by circle C.

$$\Omega = 2\pi(1-\cos\alpha)$$

Note: If the surface for previous problem is hemisphere then solid angle subtended by hemisphere at its centre O.

$$\Omega = 2\pi\left(1-\cos\frac{\pi}{2}\right) \quad \left(\theta = \frac{\pi}{2} \text{ for hemisphere}\right)$$

$$= 2\pi$$

For sphere : $\theta = \pi$

So, Solid angle subtended sphere at its centre

$$\Omega = 2\pi(1-\cos\pi)$$

$$= 4\pi$$

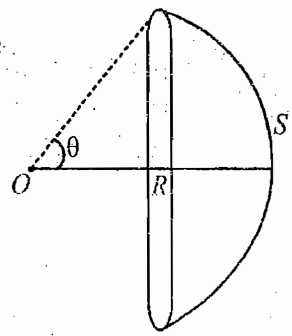


Figure 7.56

EXERCISE

1. Verify Gauss Divergence theorem for the vector $\vec{F} = x^2\hat{i}+y^2\hat{j}+z^2\hat{k}$ taken over the cube $0 \le x, y, z, \le 1$.
2. If $\vec{u} = 4y\hat{i}+x\hat{j}+2z\hat{k}$,calculate using Gauss Divergence theorem the surface integral $\int_S \nabla\times\vec{u}.\hat{n}dS$ over the hemisphere given by $x^2+y^2+z^2=a^2, z\ge 0$. **Ans.** $-3\pi a^2$
3. Verify Gauss divergence theorem to show that $\oint((x^3-yz)\hat{i}-2x^2\hat{j}+2\hat{k}).\hat{n}dS = \frac{1}{3}a^5$ where S denotes the surface of the cube bounded by the planes $x=0, x=a, y=0, y=a, z=0, z=a$.

4. Evaluate $\oint(x\hat{i}+y\hat{j}+z\hat{k}).\hat{n}dS$ where S denotes the surface of cube bounded by planer $x=0, x=a, y=0, y=a, z=0, z=a$ by application of Gauss divergence theorem Verify your answer by evaluating the integral directly. **Ans.** a^3

5. Evaluate $\oint(ax^2+by^2+cz^2)dS$ over the sphere $x^2+y^2+z^2=r^2$ using divergence theorem.

Ans. $\frac{4}{3}\pi r^3(a+b+c)$

6. Verify divergence theorem for the function $\vec{F}=y\hat{i}+x\hat{j}+z^2\hat{k}$ over the cylindrical region bounded by $x^2+y^2=a^2, z=0, \& z=h.$

7. Evaluate $\iint_S x^3dydz+x^2ydzdx+x^2z\,dxdy$ by Gauss Divergence theorem, where S is the surface of the cylinder $x^2+y^2=a^2$ bounded by $z=0$ and $z=b$. **Ans.** $\frac{5}{4}\pi a^4b$

8. Evaluate $\oint\vec{F}.\hat{n}dS$ where $\vec{F}=2xy\hat{i}+yz^2\hat{j}+xz\hat{k}$ and S is the surface of the parallelopiped bounded by $x=0, y=0, z=0, x=2, y=1$ and $z=3$. **Ans.** 30

9. Use Gauss Theorem of evaluate $\iint_S xdydz+ydzdx+zdxdy$ S being the surface of cube bounded by $x=0, x=a, y=0, y=a, z=0, z=a.$ **Ans.** $3a^3$

10. Use Gauss theorem to evaluate $\oint\vec{F}.\hat{n}dS$ where $\vec{F}=(x^3-yz)\hat{i}-2x^2y\hat{j}+2\hat{k}$. S is the cube bounded by $x=0, x=a, y=0, y=a, z=0, z=a.$ **Ans.** $\frac{1}{3}a^5$

11. Use Gauss theorem to evaluate $\iint_S y^2z^2dydz+z^2x^2dzdx+(x^2+y^2)dxdy$ where S is the surface of sphere $x^2+y^2+z^2=a^2$ above the xy plane. **Ans.** 2

12. Use Gauss theorem to evaluate $\iint_S x^2dydz+y^2dzdx+z^2dxdy$ where S is the surface of ellipsoid $ax^2+by^2+cz^2=1.$ **Ans.** 0

13. Using Gauss theorem, evaluate $\oint\vec{F}.\hat{n}dS$ where $\vec{F}=x^3\hat{i}+y^3\hat{j}+z^3\hat{k}$ and S is the surface of sphere of radius 1 and centre at origin. **Ans.** $\frac{12\pi}{5}$

14. Evaluate using Gauss theorem $\iint_S xzdxdy+xydydz+yzdzdx$ where S is pyramid formed by planer $x=0, y=0, z=0, x+y+z=a.$ **Ans.** $\frac{a^3}{8}$

15. Using Gauss Divergence theorem, evaluate $\iint_S xzdydz+xydzdy+xydxdy$ where S is the surface located in the first octant and formed by cylinder $x^2+y^2=1$ and planes $x=0, y=0, z=0, z=2.$

Ans. $\left(\frac{4}{3}+\frac{\pi}{2}\right)$

16. If $\vec{F}=2x^2\hat{i}-4yz\hat{j}+zx\hat{k}$. Evaluate $\oint\vec{F}.\hat{n}dS$ where is surface of the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1.$ **Ans.** $\frac{1}{2}$

8

STOKE'S THEOREM

Let S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. Let $\vec{F}(x, y, z)$ be a continuous vector function which has continuous first partial derivatives in a region of space which contains S in its interior. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_S \nabla \times \vec{F} \cdot \hat{n}\, dS$$

where C is traversed in the positive direction. The direction of C is called positive if an observer, walking on the boundary of S in this direction with his head pointing in the direction of outward drawn normal $\hat{n}$ to S, has surface on the left. In other words, the line integral of the tangential component of vector $\vec{F}$ taken around a simple closed curve C is equal to the surface integral of the normal component of curl of $\vec{F}$ taken over any surface S having C as its boundary.

CARTESIAN FORM OF STOKE'S LAW

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k},$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

The outward drawn normal to S, $\hat{n}$ can be written in terms of directional cosines as

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

α, β, γ are the angles with $\hat{n}$ makes with x, y, z axes respectively.

So, $$\nabla \times F.\hat{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma$$

Also, $$\vec{F}.d\vec{r} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}).(dx\hat{i} + dy\hat{j} + dz\,\hat{k})$$
$$= F_1dx + F_2dy + F_3dz$$

So, stoke's can be written in cartesian form as

$$\oint_C F_1dx + F_2dy + F_3dz = \int_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS$$

Proof : Let S be the surface whose projection on xy, yz & xz plane are bounded by simple closed curves. The equation of S can be written as.

	$z = f(x, y)$	(z as a explicit function of x & y)
or,	$x = f(y, z)$	(z as a explicit function of y & z)
or,	$y = h(z, x)$	(y as a explicit function of z & x)

$$\int_S \nabla\times\vec{F}.\hat{n}dS = \int_S \nabla\times(F_1\hat{i}+F_2\hat{j}+F_3\hat{k}).\hat{n}dS$$

$$= \int_S \nabla\times(F_1\hat{i}).\hat{n}dS + \int_S \nabla\times(F_2\hat{j}).\hat{n}dS + \int_S \nabla\times(F_3\hat{k}).\hat{n}dS$$

$$\nabla\times F_1\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix} = \frac{\partial F_1}{\partial z}\hat{j} - \frac{\partial F_1}{\partial y}\hat{k}$$

So, $$\int_S \nabla\times F_1\hat{i}.\hat{n}dS = \int_S \left(\frac{\partial F_1}{\partial z}\hat{j}.\hat{n} - \frac{\partial F_1}{\partial y}\hat{k}.\hat{n}\right)dS \qquad \text{....(1)}$$

Now, we will show that the above surface integral is equal to $\oint_C F_1 dx$.

Let R be the orthogonal projection of S on xy plane and Γ be its boundary which is oriented as shown in Figure 8.1 using representation $z = f(x,y)$ on S we can write.

$$\oint_C F_1(x,y,z)dx = \int_\Gamma F_1(x,y,f(x,y))dx$$

$$= \int_\Gamma F_1(x,y,f(x,y))dx + 0.dy$$

$$= -\iint_R \frac{\partial F_1}{\partial y}.dxdy$$

$$\left(\text{using Green's theorem, } \int_S Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy\right)$$

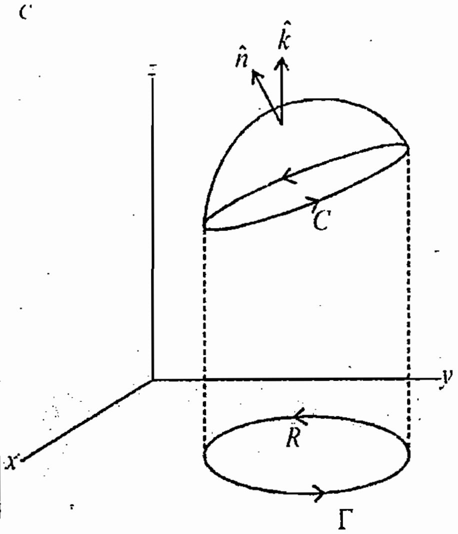


Figure 8.1

$$\oint F_1(x,y,z)dx = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F}{\partial z}\cdot\frac{\partial f}{\partial y}\right)dxdy \qquad \text{....(2)}$$

$$\left(\frac{\partial F_1}{\partial y}(x,y,f(x,y)) = \frac{\partial F_1}{\partial y} + \frac{\partial F}{\partial z}\cdot\frac{\partial f}{\partial y}\right)$$

The surface $z = f(x, y)$ belongs to the family of level surfaces given by

$$S : z - f(x,y) = \text{constant}$$

A unit normal to the surface S is given by

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{1}{|\nabla S|}\left[-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}\right]$$

$$\hat{j}.\hat{n} = -\frac{1}{|\nabla S|}\frac{\partial f}{\partial y} \text{ and } \hat{k}.\hat{n} = \frac{1}{|\nabla S|}$$

$$dS = \frac{dxdy}{|\hat{n}.\hat{k}|} = |\nabla S| dxdy$$

So, from (1)

$$\int_S \nabla \times (F_1\hat{i}).\hat{n}dS = -\iint\left(\frac{\partial F_1}{\partial z}\cdot\left(-\frac{1}{|\nabla S|}\frac{\partial f}{\partial y}\right) - \frac{\partial F_1}{\partial y}\cdot\frac{1}{|\nabla S|}\right)|\nabla S|\, dxdy$$

$$= -\iint\left(\frac{\partial F_1}{\partial z}\cdot\frac{\partial f}{\partial y} + \frac{\partial F_1}{\partial y}\right)dxdy \qquad ....(3)$$

From (2) & (3)

$$\int_S \nabla \times (F_1\hat{i}).\hat{n}dS = \oint_C F_1 dx$$

Similarly,

$$\int \nabla \times (F_2\hat{j}).\hat{n}dS = \oint F_2 dy$$

$$\int \nabla \times (F_3\hat{k}).\hat{n}dS = \oint F_3 dz$$

Hence, $\oint_C \vec{F}.d\vec{r} = \oint_S \nabla \times \vec{F}.\hat{n}dS$

If S is a piecewise smooth surface consisting of n surface $S_1, S_2, ... S_n$ with boundaries $C_1, C_2, ... C_n$ then, Stoke's law hold for each such surface intergral; then sum of surfaces over $S_1, S_2, ... C_n$ will give us surface integral over S while the sum of integrals over $C_1, C_2 .... C_n$ will gives us line integral over C.

Corollary : If R is the region in xy plane bounded by closed curve C. Let the vector function $\vec{F} = M\hat{i} + N\hat{j}$, $\hat{n} = \hat{k}, dS = dxdy$

$$\int_R \nabla \times \vec{F}\cdot\hat{n}\, dS = \iint \nabla \times \vec{F}\cdot\hat{k}\, dxdy = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

So, by Stoke's theorem

$$\oint \vec{F}\cdot d\vec{r} = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

This is Green's theorem in plane.

SOLVED EXAMPLES (OBJECTIVE)

1. The value of $\oint \vec{r}\cdot d\vec{r}$ is equal to

(a) 0 (b) 2 (c) 3 (d) 1

Ans. (a)

Using Stoke's law

$$\oint_C \vec{r}\cdot d\vec{r} = \oint_S \nabla \times \vec{r}\cdot\hat{n}\, dS$$

$$= 0 \qquad \text{as } (\nabla \times \vec{r} = 0)$$

2. The value of $\oint \phi\nabla\psi\cdot d\vec{r} + \oint \psi\nabla\phi\cdot d\vec{r}$

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (a)

$$\oint_C \phi\nabla\psi\cdot d\vec{r} + \oint \psi\nabla\phi\cdot d\vec{r} = \oint_C (\phi\nabla\psi + \psi\nabla\phi)\cdot d\vec{r}$$

$$= \int_C \nabla(\phi\psi)\cdot d\vec{r}$$

$$= \int_S \nabla\times(\nabla\phi\psi)\cdot\hat{n}\,dS \qquad \text{(By Stoke's theorem } \oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS\text{)}$$

$$= 0 \qquad \text{(as Curl of gradient of scalar function = 0)}$$

3. The value of $\oint_C \phi\nabla\phi\cdot d\vec{r}$ for closed curve C is equal to

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (a)

$$\oint_C \phi\nabla\psi\cdot d\vec{r} = \int_S \nabla\times(\phi\nabla\phi)\cdot\hat{n}\,dS \qquad \text{(Using Stoke's law)}$$

$$= \int_S (\nabla\phi\times\nabla\phi + \phi\nabla\times\nabla\phi)\cdot\hat{n}\,dS = 0 \quad (\text{as } \nabla\phi\times\nabla\phi = 0 \ \&\ \nabla\times(\nabla\phi) = 0)$$

SOLVED EXAMPLES (SUBJECTIVE)

1. Prove that $\oint \phi\,d\vec{r} = \int_S \overline{dS}\times\nabla\phi$.

Solution.

Let $\vec{C}$ be an arbitrary constant vector

Let $\vec{F} = \phi\vec{C}$

By Stoke's theorem

$$\oint_C \phi\vec{C}\cdot d\vec{r} = \int_S \nabla\times(\phi\vec{C})\cdot\overline{dS}$$

$$= 0 + \int_S (\overline{dS}\times\nabla\phi)\cdot\vec{C}$$

$$= \vec{C}\cdot\int_S \overline{dS}\times\nabla\phi$$

So, $$\vec{C}\cdot\oint_C \phi\,d\vec{r} = \vec{C}\cdot\int_S \overline{dS}\times\nabla\phi$$

Hence, $$\int_C \phi\,d\vec{r} = \int_S \overline{dS}\times\nabla\phi$$

2. By Stoke's theorem, prove that curl grad $\phi = 0$

Solution.

Let S be any open surface bounded by a closed curve C as shown in fig. 8.2. Let vector function be

$$\vec{F} = \text{grad}\,\phi$$

By Stoke's theorem

$$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$$

If $\vec{F} = \nabla\phi$

$$\Rightarrow \int \nabla\times(\nabla\phi)\cdot\hat{n}\,dS = \oint \nabla\phi\cdot d\vec{r}$$

$$= \oint d\phi = 0$$

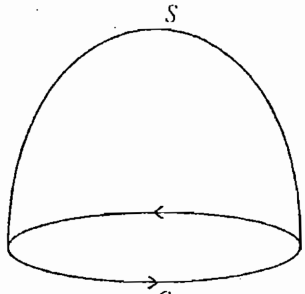


Figure 8.2

Since, surface integral is zero for any arbitrary surface S. So, the integrand has to be zero.

So, $\quad$ Curl grad $\phi = 0$

3. Verify Stokes theorem for $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = a^2$ and C is its bounding curve.

Solution.

S is the surface of sphere $x^2 + y^2 + z^2 = a^2$ lying above xy plane and bounded by the circle $C : x^2 + y^2 = a^2$

On curve C, $x = a\cos\theta, y = a\sin\theta,\ z = 0$

$dx = -a\sin\theta d\theta, dy = a\cos\theta d\theta,\ dz = 0$

So,
$$\vec{F}\cdot d\vec{r} = ydx + zdy + xdz$$
$$= ydx \qquad (z = 0 \text{ on } C)$$
$$\oint \vec{F}\cdot d\vec{r} = \int ydx$$
$$= -\int_0^{2\pi} a\sin\theta \cdot a\sin\theta d\theta$$
$$= -a^2 \int_0^{2\pi} \sin^2\theta d\theta = -\pi a^2$$

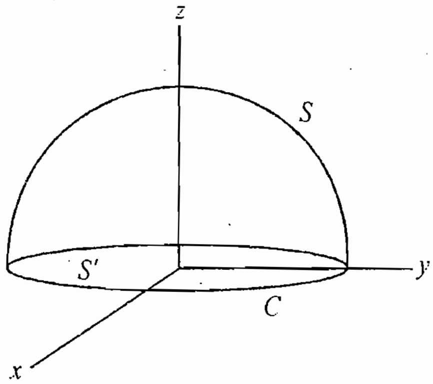


Figure 8.3

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$$

Consider a closed piecewise smooth surface Σ consisting of spherical surface $S : x^2 + y^2 + z^2 = a^2$ and $S' : z = 0$ [illegible] a volume V.

$$\oint_\Sigma \nabla\times\vec{F}\cdot\hat{n}\, dS = \int_V \nabla\cdot(\nabla\times\vec{F})dS = 0$$

$$\Rightarrow \quad \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS + \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS = 0$$

$$\Rightarrow \quad \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$

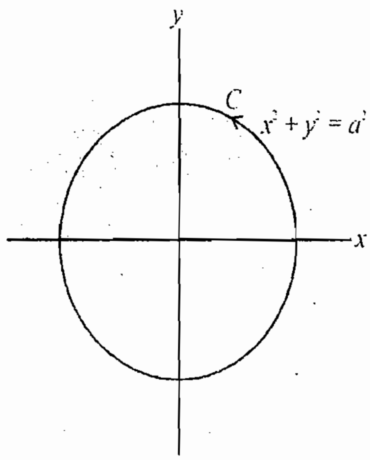


Figure 8.4

On S, outward drawn unit normal $\hat{n} = -\hat{k}$

$$\nabla\times\vec{F}\cdot\hat{n} = (-\hat{i} - \hat{j} - \hat{k})\cdot(-\hat{k}) = 1$$

So,
$$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$
$$= -\int_{S'} dS$$
$$= -\pi a^2$$

Hence,
$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$$

Stoke's theorem is verified.

4. Verify Stoke's theorem for
$$\vec{F} = (x^2 + y^2)\hat{i} - 2xy\,\hat{j}$$
taken round the rectangle bounded by $x = \pm a, y = 0, y = b$

Solution.

C is a piecewise smooth curve consisting of $y = 0, x = a, y = b$ & $x = -a$. The curve C encloses a plane surface lying in xy plane as shown in fig. 8.5.

Let us first evaluate the surface integral $\int_S \nabla \times \vec{F}.\hat{n}\, dS$.

Let us orient the curve in anticlockwise direction. With this orientation $\hat{n} = \hat{k}$.

You must remember that the direction of C is called positive if an observer walking on the boundary of S in this direction with his head in the direction of outward drawn normal $\hat{n}$ to S, has surface on his left.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= -4y\hat{k}$$

$$\nabla \times \vec{F} \cdot \hat{n} = -4y$$

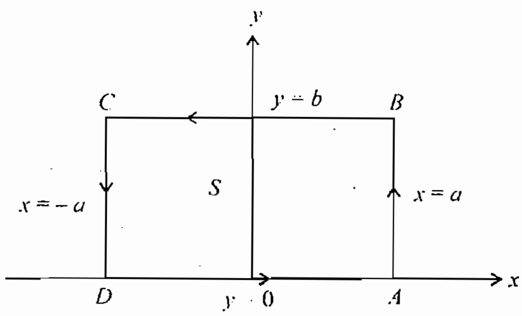


Figure 8.5

The surface element $dS = dxdy$

$$\int_S \nabla \times \vec{F} \cdot \hat{n}\, dS = -4\int_{-a}^{a}\int_0^b y\, dydx$$

$$= -4ab^2.$$

Now, Let us evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r}$$

$$\vec{F} \cdot d\vec{r} = (x^2 + y^2)dx - 2xydy$$

On AB, $x = a$, $dx = 0$, y varies from 0 to b,

$$\vec{F} \cdot d\vec{r} = -2aydy$$

$$\int_{AB} \vec{F} \cdot d\vec{r} = -2a\int_0^b y\, dy = -ab^2$$

On BC, $y = b$, $dy = 0$, x varies from a to $-a$

$$\vec{F} \cdot d\vec{r} = (x^2 + b^2)\, dx$$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^{-a} (x^2 + b^2)\, dx = \frac{x^3}{3} + b^2 x \Big|_a^{-a}$$

$$= -\frac{2a^3}{3} - 2ab^2$$

On CD, $x = -a$, $dx = 0$, y varies from b to 0.

$$\vec{F} \cdot d\vec{r} = 2aydy$$

$$\int_{CD} \vec{F} \cdot d\vec{r} = 2a\int_b^0 y\, dy = -ab^2$$

On DA, $y = 0$, $dy = 0$, x varies from $-a$ to a.

$$\vec{F} \cdot d\vec{r} = x^2 dx$$

$$\int_{DA} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2 dx = \frac{2a^3}{3}$$

So, $$\oint_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}$$

$$= -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$= -4ab^2$$

Hence, $$\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$$

5. Verify Stoke's theorem for $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=a^2$ and C is its boundary.

Solution.

The surface S is the part of sphere $x^2+y^2+z^2=a^2$ above xy plane bounded by curve C, $x^2+y^2=a^2, z=0$ as shown in fig. 8.6.

On curve C, $x = a\cos\theta, y = a\sin\theta, z = 0$

$$dx = -a\sin\theta\, d\theta,\ dy = a\cos\theta\, d\theta,\ dz = 0$$

$$\vec{F}\cdot d\vec{r} = (2x-y)dx - yz^2dy - y^2zdz$$

$$= (2a\cos\theta - a\sin\theta)(-a\sin\theta\, d\theta)$$

$$= a^2(\sin^2\theta - 2\sin\theta\cos\theta)d\theta$$

So, $$\oint_C \vec{F}\cdot d\vec{r} = a^2\int_0^{2\pi}(\sin^2\theta - 2\sin\theta\cos\theta)d\theta$$

$$= \pi a^2$$

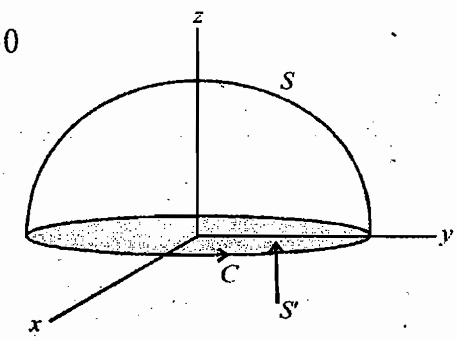


Figure 8.6

Now, let us evaluate surface integral $\int_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{k}$$

Now, consider a closed piecewise smooth surface Σ consisting of spherical part $S: x^2+y^2+z^2=a^2$ and its base, $S': z=0$ as shown in fig. 8.6.

By Gauss divergence theorem

$$\oint_\Sigma \nabla\times\vec{F}\cdot\hat{n}\,dS = \int_V \nabla\cdot(\nabla\times\vec{F})d\tau = 0 \quad (V \text{ is volume enclosed by } \Sigma)$$

$$\Rightarrow \quad \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS + \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS = 0$$

So, $$\int \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$

On S', outward drawn unit normal vector $\hat{n} = -\hat{k}$

$dS = dxdy$, $\nabla\times\vec{F}\cdot\hat{n} = (\hat{k})\cdot(-\hat{k}) = -1$

So, $$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$

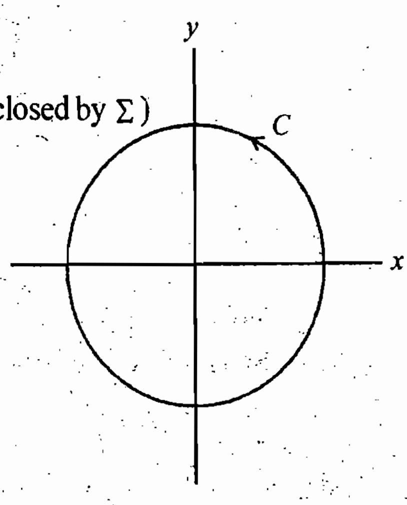


Figure 8.7

$$= \int_{S'} dS = \text{area of base} = \pi a^2$$

So, $$\oint \vec{F}\cdot d\vec{r} = \oint_S \nabla\times\vec{F}\cdot n\,dS$$

Hence, Stoke's theorem is verified.

6. Verify Stoke's theorem for $\vec{F} = -y^3\hat{i} + x^3\hat{j}$ where S is the circular disc $x^2+y^2 \le a^2, z = 0$.

Solution.

S is the circular disc bounded by circle $x^2+y^2=a^2, z=0$ (curve C) as shown in fig.8.8.

On curve C, $x = a\cos\theta$, $dx = -a\sin\theta\, d\theta$

$$y = a\sin\theta,\ dy = a\cos\theta\, d\theta$$

$$\vec{F}\cdot d\vec{r} = -y^3dx + x^3dy = (a^4\sin^4\theta + a^4\cos^4\theta)d\theta$$

So, $$\oint_C \vec{F}\cdot d\vec{r} = a^4\int_0^{2\pi}(\sin^4\theta + \cos^4\theta)\,d\theta$$

$$= 4a^4\left[\frac{\Gamma(5/2)\Gamma(1/2)}{2\Gamma(3)} + \frac{\Gamma(1/2)\Gamma(5/2)}{2\Gamma(3)}\right] = \frac{3\pi}{2}a^4$$

Now, let us evaluate, the surface integral $\int_S \nabla\times\vec{F}.\hat{n}dS$

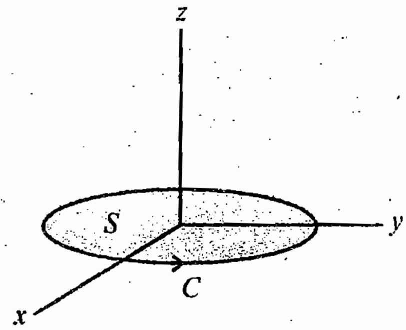


Figure 8.8

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= 3(x^2+y^2)\,\hat{k}$$

Outward drawn unit normal vector to S, $\hat{n} = \hat{k}$

$x = r\cos\theta$, $y = r\sin\theta$

$$\nabla\times\vec{F}\cdot\hat{n} = 3(x^2+y^2) = 3r^2.$$

$$dS = dxdy = rdrd\theta$$

$$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = 3\int_0^{2\pi}\int_0^a r^3\cdot dr\,d\theta$$

$$= \frac{3}{4}a^4\int_0^{2\pi} d\theta$$

$$= \frac{3}{2}\pi a^4$$

Thus, $$\oint_S \vec{F}\cdot d\vec{r} = \int_{S'} \nabla\times\vec{F}\cdot n\,dS$$

Thus, Stoke's theorem is verified.

7. Verify Stoke's theorem for the function

$$\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$$

where curve C is the unit circle in the xy plane bounding the hemisphere $z = \sqrt{1-x^2-y^2}$.

Solution.

The curve C is a unit circle $x^2+y^2=1, z=0$ bounding the surface S which is a hemisphere of unit radius given by $z=\sqrt{1-x^2-y^2}$.

On $C, x=\cos\theta, y=\sin\theta, z=0$

$dx=-\sin\theta d\theta, dy=\cos\theta d\theta, dz=0$

$$\vec{F}\cdot d\vec{r} = zdx+xdy+ydz$$
$$= xdy \text{ on curve } C$$

So, $$\oint \vec{F}\cdot d\vec{r} = \int_C x\,dy$$
$$= \int_0^{2\pi} \cos^2\theta d\theta = \pi$$

Now, let us evaluate the surface integral $\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$.

Consider a closed piecewise smooth surface consisting of hemisphere S and base S' as shown in figure 8.9.

By Gauss Divergence theorem

$$\oint_\Sigma \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \int_V \nabla\cdot(\text{curl}\,\vec{F})\cdot\hat{n}\,dS = 0$$

$$\Rightarrow \quad \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS + \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS = 0$$

So, $$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$

On $S', \hat{n}=-\hat{k}, dS=dxdy$,

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$
$$= \hat{i}+\hat{j}+\hat{k}$$

On S', $\quad \nabla\times\vec{F}\cdot\hat{n} = -1$

So, $$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$
$$= \int_{S'} dS = \text{area of base } S'$$
$$= \pi$$

Hence, $$\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$$

Thus, Stoke's theorem is verified.

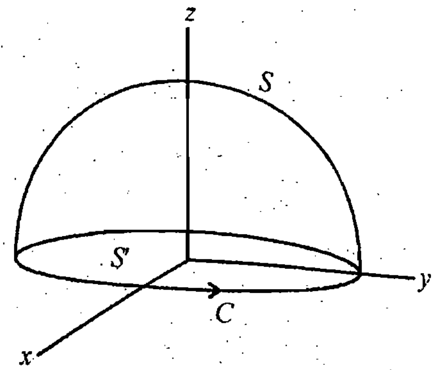


Figure 8.9

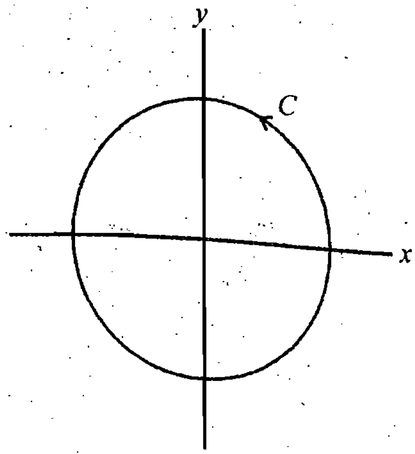


Figure 8.10

8. Evaluate by Stoke's theorem

$$\oint e^x dx + 2y dy - dz$$

where C is the curve $x^2+y^2=9$ & $z=2$.

Solution.

The curve C is a circle of radius 3 units at a height 3 units from xy plane and having centre on the z axis. Let the surface enclosed by this curve is a disc of radius 3 as shown in fig. 8.11. Students kindly note that the man with head in the direction of $\hat{n}$ and moving along the given orientation along the perifery should see the surface on his left. So, the direction of $\hat{n}$ and orientation should be matched accordingly

$$\oint e^x dx + 2y dy - dz = \oint (e^x \hat{i} + 2y\hat{j} - \hat{k}) \cdot d\vec{r}$$

So, $\vec{F} = e^x \hat{i} + 2y\hat{j} - \hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = 0$$

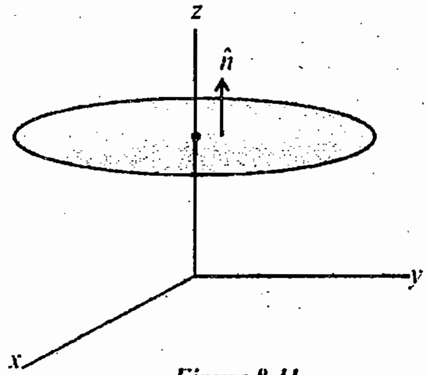


Figure 8.11

By Stokes theorem

$$\oint \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot \hat{n}\, dS$$

$$= 0 \qquad (\text{Since, curl } \vec{F} = 0)$$

9. Evaluate by Stoke's theorem

$$\oint yz dx + xz dy + xy dz$$

where C is the curve of intersection of $x^2 + y^2 = 1, z = y^2$.

Solution.

The curve C is the curve of intersection of $x^2 + y^2 = 1$ and $x = y^2$ as shown in fig 8.12.

$$\oint yz dx + xz dy + xy dz = \oint (yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}) \cdot d\vec{r}$$

where $\vec{F} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = 0$$

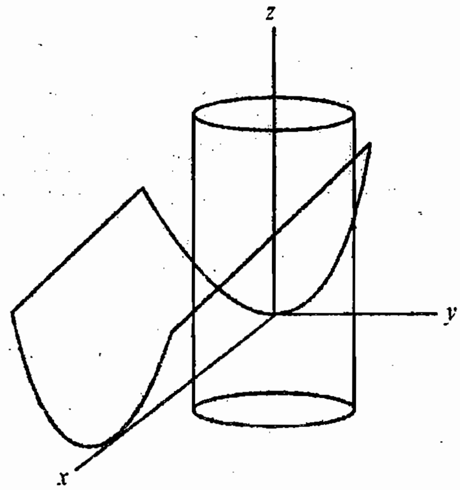


Figure 8.12

By Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot \hat{n}\, dS$$

$$= 0 \qquad (\text{as curl } \vec{F} = 0)$$

10. Evaluate $\oint_C xy dx + xy^2 dy$ by Stoke's theorem where C is the square in xy plane with vertices $(a, 0)$, $(-a, 0)$, $(0, a)$, $(0, -a)$.

Solution.

The curve C is a square enclosing a plane surface S is xy plane.

The normal to S for anticlockwise orientation is $\hat{n} = \hat{k}$

$$\oint xy dx + xy^2 dy = \oint (xy\hat{i} + xy^2\hat{j}) \cdot d\vec{r}$$

where $\vec{F} = xy\hat{i} + xy^2\hat{j}$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

$$= (y^2 - x)\hat{k}$$

So, $\nabla\times\vec{F}\cdot\hat{n} = (y^2 - x)$

$$dS = dxdy$$

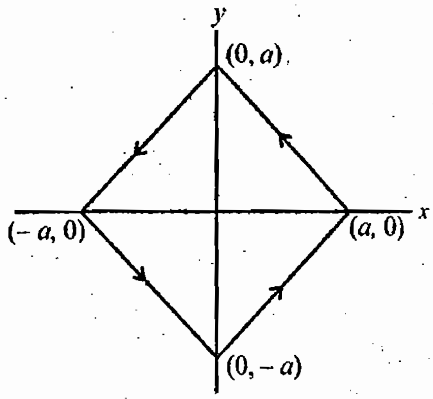


Figure 8.13

By Stoke's theorem

$$\oint \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$$

$$= \iint (y^2 - x)\,dxdy$$

The curve C consists of $x + y = a$, $x + y = -a$, $x - y = a$, $x - y = -a$.

To solve double integral

Let $u = x + y$

$$v = x - y$$

So, $x = \dfrac{u+v}{2},\ y = \dfrac{u-v}{2}$

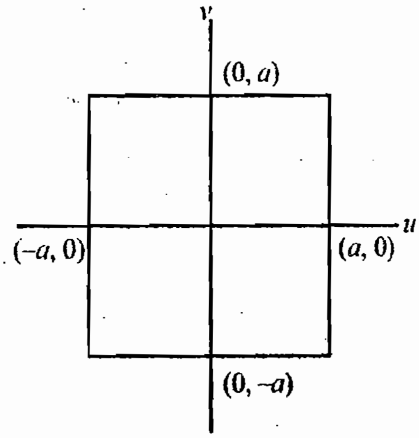


Figure 8.14

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

So, $dxdy = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| dudv$

$$= \frac{1}{2}dudv$$

$$y^2 - x = \frac{1}{4}(u-v)^2 - \left(\frac{u+v}{2}\right)$$

$$= \frac{1}{4}(u^2 + v^2 - 2uv - 2u - 2v)$$

So, $\iint\limits_S (y^2 - x)\,dxdy = \dfrac{1}{4}\int_{-a}^{a}\int_{-a}^{a}(u^2 + v^2 - 2uv - 2u - 2v)\,dudv$

$$= \frac{1}{4}\int_{-a}^{a}\left[\frac{u^3}{3} + uv^2 - u^2v - u^2 - 2uv\right]_{-a}^{a} dv$$

$$= \frac{1}{4}\int_{-a}^{a}\left(\frac{2}{3}a^3 + 2av^2 - 0 - 0 - 4av\right)dv$$

$$= \frac{1}{4}\left[\frac{2}{3}a^3 v + \frac{2a}{3}v^3 - 2av^2\right]_{-a}^{a}$$

$$= \frac{1}{4}\left[\frac{4}{3}a^4 + \frac{4a^4}{3}\right] = \frac{2}{3}a^4.$$

11. Evaluate by using Stoke's theorem, $\oint \vec{F}\cdot d\vec{r}$ where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+2z)\hat{k}$ where C is the boundary of the triangle with vertices at $(0,0,0)$, $(a,0,0)$, $(a,a,0)$.

Solution.

The curve C is triangle with vertices at $(0,0,0)$, $(a,0,0)$ & $(a,a,0)$. Let S is the surface enclosed by C in xy plane

$$\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+2z)\hat{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+2z) \end{vmatrix}$$

$$= \hat{j} + 2(x-y)\hat{k}$$

For given orientation of C, normal to S is $\hat{n} = \hat{k}$.

$$\nabla\times\vec{F}\cdot\hat{n} = 2(x-y),\ dS = dxdy$$

By Stoke's theorem

$$\oint \vec{F}\cdot d\vec{r} = \int \nabla\times\vec{F}\cdot\hat{n}\,dS$$

$$= 2\int_0^a\int_y^a (x-y)\,dxdy$$

$$= 2\int_0^a \left|\frac{x^2}{2} - xy\right|_y^a dy$$

$$= \int_0^a (a^2-y^2) - 2y(a-y)\,dy$$

$$= \int_0^a (y^2 - 2ay + a^2)\,dy$$

$$= \left|\frac{y^3}{3} - ay^2 + a^2 y\right|_0^a = \frac{a^3}{3}$$

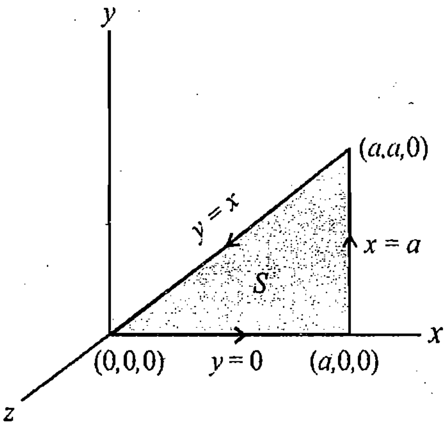


Figure 8.15

12. Evaluate by Stoke's theorem

$$\oint \sin z\,dx - \cos x\,dy + \sin y\,dz$$

where C is the boundary of the rectangle, $0 \le x \le \pi, 0 \le y \le 2, z = 4$.

Solution.

The given line integral

$$\oint \sin z\,dx - \cos x\,dy + \sin y\,dz = \oint (\sin z\,\hat{i} - \cos x\,\hat{j} + \sin y\,\hat{k})\cdot d\vec{r}$$

So, $$\vec{F} = \sin z\,\hat{i} - \cos x\,\hat{j} + \sin y\,\hat{k}$$

For given orientation of curve C, $\hat{n} = \hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}$$

$$= \cos y\,\hat{i} + \cos z\,\hat{j} + \sin x\,\hat{k}$$

$$\text{curl}\,\vec{F}\cdot\hat{n} = \sin x,\ dS = dxdy$$

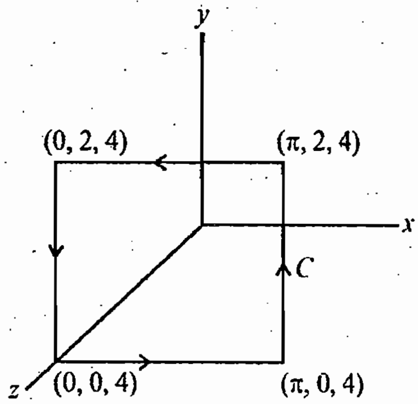


Figure 8.16

So, by Stoke's theorem

$$\oint \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$$

$$= \int_0^2\int_0^\pi \sin x\,dxdy$$

$$= 2\int_0^2 [-\cos x]_0^\pi\,dy$$

$$= 2\int_0^2 dy = 4$$

13. Let S be the surface $x^2+y^2+z^2=1,\ z\ge 0$. Use Stoke's theorem to evaluate

$$\int_C (2x-y)dx - ydy - zdz$$

where C is the circle $x^2+y^2=1, z=0$, oriented anticlockwise.

Solution.

$$\oint (2x-y)dx - ydy - zdz = \oint \vec{F}\cdot d\vec{r}$$

Here, $\vec{F} = (2x-y)\hat{i} - y\hat{j} - z\hat{k}$

By Stoke's theorem

$$\oint \vec{F}\cdot d\vec{r} = \int_S (\nabla\times\vec{F})\cdot\hat{n}dS$$

Now, we have to evaluate $\int_S (\nabla\times\vec{F})\cdot\hat{n}dS$

S is part of sphere $x^2+y^2+z^2=1$ above xy plane. Let us consider surface S made up two piecewise smooth surfaces S and S' where S' is base of hemisphere bounded by curve $x^2+y^2=1, z=0$.

By divergence theorem

$$\oint_\Sigma \nabla\times\vec{F}\cdot\hat{n}dS = \int_V \nabla.(\nabla\times\vec{F})d\tau = 0$$

$$\oint_\Sigma \nabla\times\vec{F}\cdot\hat{n}dS = \int_S \nabla\times\vec{F}\cdot\hat{n}dS + \int_{S'} \nabla\times\vec{F}\cdot\hat{n}dS = 0$$

$$\Rightarrow \quad \int_S \nabla\times\vec{F}\cdot\hat{n}dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}dS$$

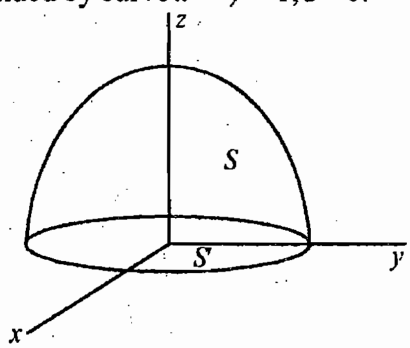


Figure 8.17

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -y & -z \end{vmatrix} = \hat{k}$$

For $S', \hat{n} = -\hat{k}$, $dS = dxdy$

$$\oint \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}dS$$

$$= \iint dxdy = \text{Area of base} = \pi$$

Hence, by stoke's theorem

$$\oint \vec{F}\cdot d\vec{r} = \pi$$

14. Let C be the curve in R^3 given by $x^2 + y^2 = a^2$, $z = 0$ traced counter-clockwise, and let $\vec{F} = x^2y^3\hat{i} + \hat{j} + z\hat{k}$. Using Stokes' theorem, evaluate $\oint_C \vec{F}\cdot d\vec{r}$

Solution.

The curve C is given by $x^2 + y^2 = a^2, z = 0$ as shown in fig 8.18.

$$\vec{F} = x^2y^3\hat{i} + \hat{j} + z\hat{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = -3x^2y^2\hat{k}$$

For surface bounded by curve C, $\hat{n} = \hat{k}$.

So, $(\nabla\times\vec{F})\cdot\hat{n} = -3x^2y^2$

Using Stoke's theorem

$$\oint \vec{F}\cdot d\vec{r} = \int (\nabla\times\vec{F})\cdot\hat{n}dS$$

$$= -3\iint x^2y^2 dxdy$$

$$= -3\int_0^{2\pi}\int_0^{a} r^5\sin^2\theta\cos^2\theta\, drd\theta$$

(putting $x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta$)

$$= \frac{-3a^6}{6}\cdot\int_0^{2\pi}\sin^2\theta\cos^2\theta\, d\theta$$

$$= -2a^6\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta$$

$$= -\frac{\pi}{8}a^6 \qquad \left(\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta = \frac{\boxed{3/2}\,\boxed{3/2}}{2\boxed{3}} = \frac{\pi}{16}\right)$$

15. Verify Stokes's theorem for the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$ and the vector field $\vec{F} = (z^2 - y)\hat{i} + (x - 2yz)\hat{j} + (2xz - y^2)\hat{k}$.

Solution.

The bounding curve C of S is given by $x^2+y^2=9, z=0$

Let $x=3\cos\theta$, $y=3\sin\theta$, $z=0$.

$$\oint \bar{F}.d\bar{r} = \int_0^{2\pi} -3\sin\theta\cdot 3(-\sin\theta)d\theta + (3\cos\theta)\cdot 3\cos\theta d\theta$$

$$= 9\int_0^{2\pi} d\theta = 18\pi$$

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2-y & x-2yz & 2xz-y^2 \end{vmatrix} = 2\hat{k}$$

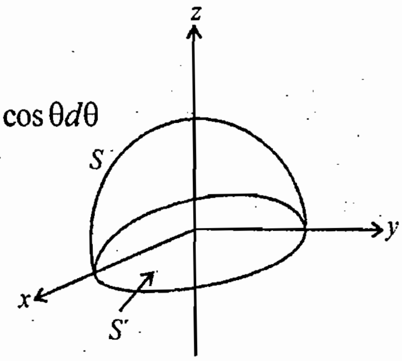


Figure 8.19

Consider a surface Σ consisting of piecewise smooth surface S & S' as shown in fig 8.19.

$$\oint \bar{F}\cdot d\bar{r} = \int_\Sigma \nabla\times\bar{F}.\hat{n}dS$$

$$= \int_S \nabla\times\bar{F}.\hat{n}dS + \int_{S'} \nabla\times\bar{F}.\hat{n}dS = 0$$

(Using Gauss Divergence theorem)

$$\int_S \nabla\times\bar{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\bar{F}.\hat{n}dS$$

$$= -\int_{S'} 2\hat{k}\cdot(-\hat{k})\,dS \quad (\text{For } S', n=-\hat{k})$$

$$= 2\int_{S'} dS$$

$$= 2\times 9\pi = 18\pi$$

So, $\oint \bar{F}.d\bar{r} = \int(\nabla\times\bar{F}).\hat{n}dS$

Hence, Stoke's theorem is verified.

16. Using Stokes theorem evaluate the line integral

$$\int_C (y\hat{i}+z\hat{j}+x\hat{k})\cdot d\bar{r}$$

where C is the intersection of $x^2+y^2+z^2=1$ and $x+y=0$ traversed in the clockwise direction when viewed from the point (1, 1, 0).

Solution.

Using stoke's theorem

$$\oint \bar{F}\cdot d\bar{r} = \int \nabla\times\bar{F}\cdot\hat{n}dS$$

$$\oint (y\hat{i}+z\hat{j}+x\hat{k})\cdot d\bar{r} = \int \nabla\times(y\hat{i}+z\hat{j}+x\hat{k})\cdot\hat{n}dS$$

$$\hat{n} = -\frac{\nabla S}{|\nabla S|} = -\frac{\hat{i}+\hat{j}}{\sqrt{2}}$$

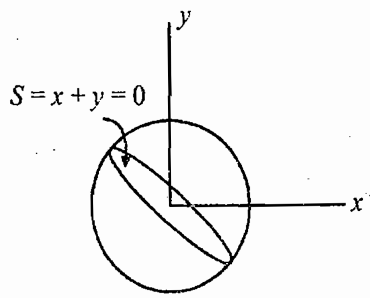


Figure 8.20

$$\nabla\times(y\hat{i}+z\hat{j}+x\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i}-\hat{j}-\hat{k}$$

$$\int(-\hat{i}-\hat{j}-\hat{k})\cdot\left(-\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)dS = \sqrt{2}\int dS = \sqrt{2}\pi$$

17. Consider a vector $\vec{A} = -4yx^2\hat{i} - 3y^2\hat{j}$

(a) Calculate the line integral $\int \vec{A}\cdot\vec{dl}$ from point $P \to O$ along the path $P \to Q \to R \to O$ as shown in the figure.

(b) Using Stoke's theorem appropriately, calculate $\int \vec{A}\cdot\vec{dl}$ for the same path $P \to Q \to R \to O$.

Solution.

$$\vec{A} = -4yx^2\hat{i} - 3y^2\hat{j}$$

$$\vec{A}\cdot\vec{dl} = -4yx^2dx - 3y^2dy$$

The line integral $\int \vec{A}\cdot\vec{dl}$ from $P \to O$ along the path $P \to Q \to R \to O$ is given by

$$\int \vec{A}\cdot\vec{dl} = \int_{PQ} \vec{A}\cdot\vec{dl} + \int_{QR} \vec{A}\cdot\vec{dl} + \int_{RO} \vec{A}\cdot\vec{dl}$$

Along PQ, $y = 1$, $dy = 0$

$$\int_{PQ} \vec{A}\cdot\vec{dl} = -4\int_0^1 x^2dx = -\frac{4}{3}$$

Along QR, $x = 1$, $dx = 0$

$$\int_{QR} \vec{A}\cdot\vec{dl} = -3\int_1^0 y^2dy = 1$$

Along RO, $y = 0$, $dy = 0$

$$\int_{RO} \vec{A}\cdot\vec{dl} = 0$$

So,
$$\int \vec{A}\cdot\vec{dl} = -\frac{4}{3} + 1 + 0 = -\frac{1}{3}$$

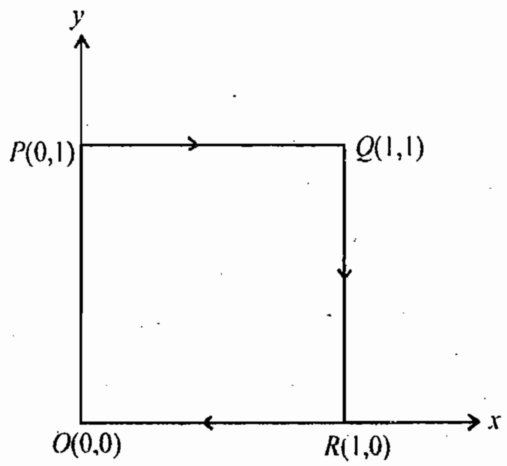


Figure 8.21

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -4yx^2 & -3y^2 & 0 \end{vmatrix} = 4x^2\hat{k}$$

For given orientation of loop

$P \to Q \to R \to O \to P$, $\hat{n} = -\hat{k}$

So,
$$\nabla\times\vec{A}\cdot\hat{n} = -4x^2$$

$$\oint \vec{A}\cdot\vec{dl} = \int \nabla\times\vec{A}\cdot\hat{n}\, dS$$

$$= \int_0^1\int_0^1 4x^2dxdy$$

$$= -\frac{4}{3}$$

Along OP, $x = 0$, $dx = 0$

$$\int_{OP} \vec{A}\cdot\vec{dl} = -\int_0^1 3y^2dy = -1$$

So, line integral along $P \to Q \to R \to O$

$$\int \vec{A}\cdot\vec{dl} = \oint \vec{A}\cdot\vec{dl} - \int_{OP} \vec{A}\cdot\vec{dl}$$

$$= -\frac{4}{3}+1 = -\frac{1}{3}$$

18. (a) Consider a constant vector field $\vec{v} = v_0\hat{k}$. Find any one of the many possible vectors $\vec{u}$, for which $\nabla\times\vec{u} = \vec{v}$.

(b) Using Stoke's theorem, evaluate the flux associated with the field $\vec{v}$ through the curved hemispherical surface defined by $x^2+y^2+z^2=r^2, z>0$.

Solution.

(a) Let

$$\vec{u} = u_x\hat{i}+u_y\hat{j}+u_z\hat{k}$$

$$\nabla\times\vec{u} = \vec{v}$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} = v_0\hat{k}$$

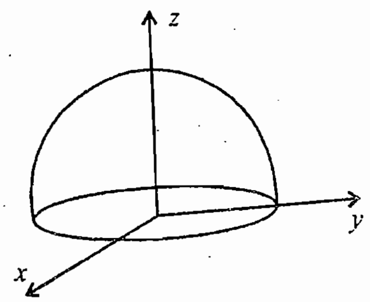


Figure 8.22

One of the possible vectors can be taken as $\vec{u} = v_0 x\,\hat{j}$

(b) Flux associated with field $\vec{v}$

$$\phi = \int_S \vec{v}\cdot\hat{n}\,dS$$

$$= \int_S \nabla\times\vec{u}\cdot\hat{n}\,dS \qquad \text{(Since } \vec{v} = \nabla\times\vec{u}\text{)}$$

$$= \oint \vec{u}\cdot d\vec{r} \qquad \text{(Using Stoke's Theorem)}$$

$$= v_0\int_C x\,dy$$

Let $x = r\cos\theta, y = r\sin\theta$

$$dy = r\cos\theta\, d\theta$$

$$\int x\,dy = r^2\int_0^{2\pi}\cos^2\theta\,d\theta = \pi r^2$$

So, $$\int_S \vec{v}\cdot\hat{n}\,dS = \pi r^2 v_0$$

19. How much work is done when an object moves from $O \to P \to Q \to R \to O$ in a force field given by

$\vec{F}(x,y) = (x^2-y^2)\hat{i}+2xy\,\hat{j}$.

Along the rectangular path shown in fig 8.23. Find the answer by evaluating the line integral and also using the Stokes' theorem.

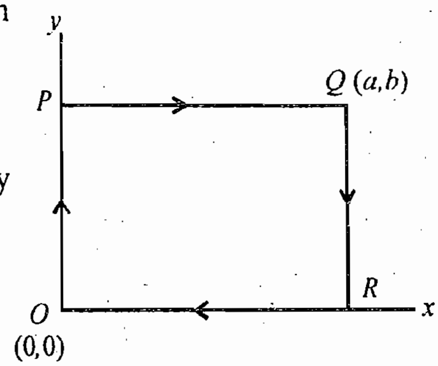


Figure 8.23

Solution.

$$\vec{F} = (x^2-y^2)\hat{i}+2xy\,\hat{j}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= (2y+2y)\hat{k} = 4y\hat{k}$$

$$\oint \vec{F}\cdot d\vec{r} = \int_{OP}\vec{F}\cdot d\vec{r} + \int_{PQ}\vec{F}\cdot d\vec{r} + \int_{QR}\vec{F}\cdot d\vec{r} + \int_{RO}\vec{F}\cdot d\vec{r} \quad ....(1)$$

For OP, $x = 0$, $dx = 0$

$$\int_{OP}\vec{F}\cdot d\vec{r} = 0$$

For PQ, $y = b$, $dy = 0$

$$\int_{PQ}\vec{F}\cdot d\vec{r} = \int_0^a (x^2-b^2)\,dx = \frac{x^3}{3} - b^2x\Big|_0^a$$

$$= \frac{a^3}{3} - ab^2$$

For QR, $x = a$, $dx = 0$

$$\int_{QR}\vec{F}\cdot d\vec{r} = \int_b^0 2ay\,dy = ay^2\Big|_b^0 = -ab^2$$

For RO, $y = 0$, $dy = 0$

$$\int \vec{F}\cdot d\vec{r} = \int_a^0 x^2\,dx = -\frac{a^3}{3}$$

Using (1)

$$\oint \vec{F}\cdot d\vec{r} = 0 + \left(\frac{a^3}{3} - ab^2\right) - ab^2 - \frac{a^3}{3}$$

$$= -2ab^2.$$

Using Stoke's theorem

$$\oint \vec{F}\cdot d\vec{r} = \int \nabla\times\vec{F}\cdot\hat{n}\,dS$$

For given orientation of loop, $\hat{n} = -\hat{k}$, $dS = dxdy$

$$\int \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_0^a\int_0^b 4y\,dxdy$$

$$= -2ab^2$$

20. A vector field is given by,

$$\vec{F}(r) = \begin{cases} \alpha(x\hat{j} - y\hat{i}) & \text{for } (x^2+y^2) \le r_0^2 \quad (\text{region} - \text{I}) \\ \alpha r_0^2 \dfrac{(x\hat{j}-y\hat{i})}{(x^2+y^2)} & \text{for } (x^2+y^2) > r_0^2 \quad (\text{region} - \text{II}) \end{cases}$$

Here α and r_0 are two constants.

(a) Find the curl of this field in both the region

(b) Find the line integral $\oint \vec{F}\cdot d\vec{l}$ along the closed semicircular path of radius $2r_0$ as shown in the figure 8.24.

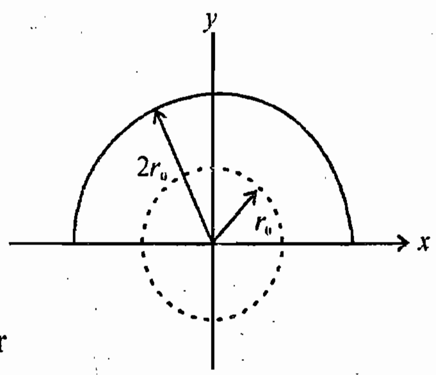


Figure 8.24

Solution.

$$\bar{F}(r) = \alpha(-y\hat{i} + x\hat{j}) \qquad \text{for } r \le r_o \quad \text{(Region I)}$$

$$= \alpha r_o^2 \frac{(-y\hat{i} + x\hat{j})}{x^2 + y^2} \qquad \text{for } r > r_o \quad \text{(Region II)}$$

(a) In regions I:

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\alpha y & \alpha x & 0 \end{vmatrix}$$

$$= 2\alpha\hat{k}$$

In region II :

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\alpha r_o^2 y}{x^2+y^2} & \frac{\alpha r_o^2 x}{x^2+y^2} & 0 \end{vmatrix} = 0$$

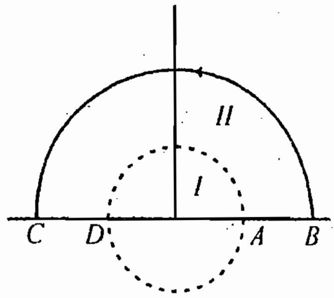


Figure 8.25

(b) Applying Stoke's law

$$\oint \bar{F} \cdot d\bar{r} = \int \nabla \times \bar{F} \cdot \hat{n}\, dS$$

$$= \int_I \nabla \times \bar{F} \cdot \hat{n}\, dS + \int_{II} \nabla \times \bar{F} \cdot \hat{n}\, dS$$

$$= \int_I \nabla \times \bar{F} \cdot \hat{n}\, dS + 0$$

$$= \int 2\alpha\, \hat{k} \cdot \hat{k}\, dS$$

$$= 2\alpha \int dS = 2\alpha\pi r_o^2 \qquad \text{(for region I, } \hat{n} = \hat{k}\text{)}$$

21. Let C be the boundary of the triangle with vertices (0,1,0), (1,0,0) and (2,1,0). If $\bar{F}(x, y, z) = -y\hat{i} + y^2 z\hat{j} + zx\hat{k}$; then use Stoke's theorem to evaluate $\int_C \bar{F}.d\bar{r}$ when C is traversed counter-clockwise when viewed from above.

Solution.

According to Stoke's Law

$$\oint \bar{F} \cdot d\bar{r} = \int_S (\nabla \times \bar{F}) \cdot \hat{n}\, dS$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & y^2 z & zx \end{vmatrix}$$

$$= -y^2\hat{i} - z\,\hat{j} + \hat{k}$$

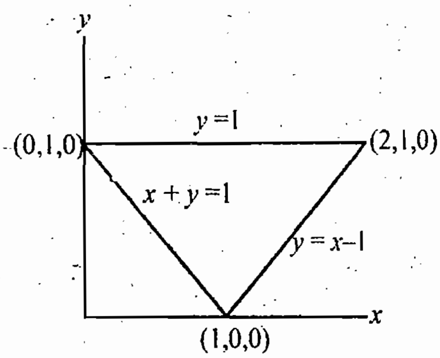


Figure 8.26

For the surface bounded by triangle, $\hat{n} = \hat{k}$.

$$(\nabla \times \bar{F}) \cdot \hat{n} = 1,\ dS = dxdy,$$

$$\int_S (\nabla\times\vec{F})\cdot\hat{n}dS = \int dS = \int_0^1\int_{1-y}^{y+1} dxdy$$

$$= \int_0^1 2ydy = 1$$

22. Verify Stokes' theorem for

$\vec{F} = (2x-3y)\hat{i} + y^2z^3\hat{j} + y^3z^2\hat{k}$,

$S: x^2+y^2+z=1, z\geq 0.$

C: the bounding curve of S.

Solution.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_C (2x-3y)\,dx + y^2z^3dy + y^3z^2dz$$

C is curve of intersection of $x^2+y^2+z=1$ and $z=0$.

$x=\cos\theta, y=\sin\theta, z=0.$

$$\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (2\cos\theta - 3\sin\theta)(-\sin\theta\, d\theta)$$

$$= \int_0^{2\pi} (3\sin^2\theta - 2\sin\theta\cos\theta)\,d\theta = 3\pi$$

Now, consider a surface Σ consisting of two piecewise smooth surfaces S & S' where S is paraboloid $x^2+y^2+z=1$ and S' is part of xy plane, $z=0$.

$$\oint_\Sigma \nabla\times\vec{F}\cdot\hat{n}dS = 0$$

$$\Rightarrow \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS + \int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS = 0$$

$$\Rightarrow \int_S \nabla\times\vec{F}\cdot\hat{n}dS = -\int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-3y & y^2z^3 & y^3z^2 \end{vmatrix}$$

$$= 3\hat{k}$$

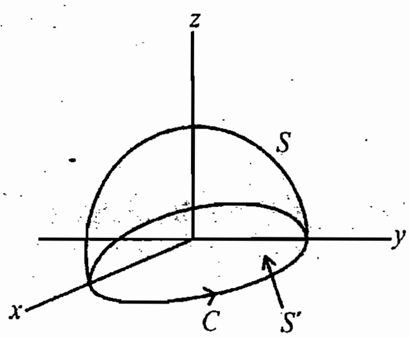


Figure 8.27

For S', $\hat{n} = -\hat{k}$

$$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS = 3\int dS = 3\pi$$

So, $\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$ (verified)

23. Verify Stokes' theorem for the function

$$\vec{F} = x^2\hat{i} - xy\hat{j}$$

integrated round the square in the plane $z=0$ and bounded by the lines $x=0, y=0, x=a$ and $y=a$, $a>0$.

Solution.

$$\vec{F} = x^2\hat{i} - xy\hat{j}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix}$$

$$= -y\hat{k}$$

$$\hat{n} = \hat{k}$$

$$\nabla\times\vec{F}\cdot\hat{n} = -y$$

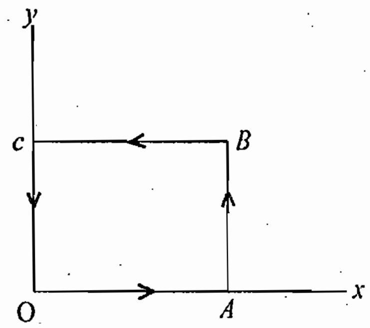


Figure 8.28

So, $$\int\nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_0^a\int_0^a y\,dx\,dy$$

$$= -\frac{a^3}{2}$$

$$\oint\vec{F}\cdot d\vec{r} = \int_{OA}\vec{F}.d\vec{r} + \int_{AB}\vec{F}.d\vec{r} + \int_{BC}\vec{F}.d\vec{r} + \int_{CO}\vec{F}.d\vec{r}$$

$$= \int_0^a x^2dx - a\int_0^a y\,dy + \int_a^0 y^2dx + \int_a^0 0\cdot dy$$

$$= \frac{a^3}{3} - \frac{a^2}{2} - a^3 = -\frac{7}{6}a^3$$

24. Apply Stoke's theorem to prove that

$$\int_C ydx + zdy + xdz = -2\sqrt{2}\,\pi a^2$$

where C is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0$, $x + y = 2a$ and begins at the point $(2a, 0, 0)$ and goes first below the xy plane.

Solution.

Curve C is the curve of intersection of sphere $x^2 + y^2 + z^2 - 2ax - 2ay = 0$ and plane $x + y = 2a$ as shown in Fig. 8.29.

The centre of sphere $(a, a, 0)$ and radius $\sqrt{2}a$.

$(x - a^2) + (y - a)^2 + c^2 = 2a^2$

The centre of sphere $(a, a, 0)$ lies on the plane $x + y = 2a$.

So, the curve of intersection is the greatest circle. Let the surface enclosed by greatest circle is a disc of radius $\sqrt{2}a$ as shown in Fig.8.29.

For given orientation of curve C, the normal to the surfaces S is $\hat{n} = \dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$.

$$\oint_C ydx + zdy + xdz = \oint(y\hat{i} + z\hat{j} + x\hat{k})\cdot d\vec{r} = \oint\vec{F}\cdot d\vec{r}$$

So, $$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$$

$$\nabla\times\bar{F}\cdot\hat{n} = -(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$$
$$= -\sqrt{2}$$

By Stoke's law

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \nabla\times\bar{F}\cdot\hat{n}\,dS$$
$$= -\sqrt{2}\int_S dS$$
$$= -\sqrt{2}\times \text{area of disc of radius } \sqrt{2}a$$
$$= -2\sqrt{2}\pi a^2$$

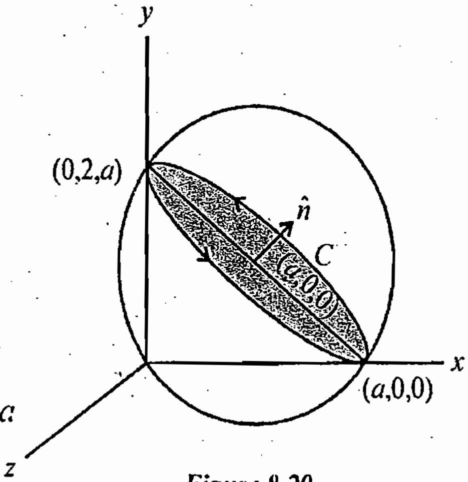


Figure 8.29

25. Use Stoke's theorem to evaluate $\int_S \nabla\times\bar{F}\cdot\hat{n}\,dS$ where $\bar{F} = y\hat{i}+(x-2xz)\hat{j}-xy\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=1$ above the xy plane.

Solution.

S is the surface of sphere $x^2+y^2+z^2=1$ above the xy plane. The boundary curve C of S is a circle $x^2+y^2=1$ lying in xy plane.

$$\bar{F} = y\hat{i}+(x-2xz)\hat{j}-xy\hat{k}$$

On C, $x=\cos\theta, dx=-\sin\theta d\theta$

$y=\sin\theta,\ dy=\cos\theta d\theta$

$z=0,\ dz=0$

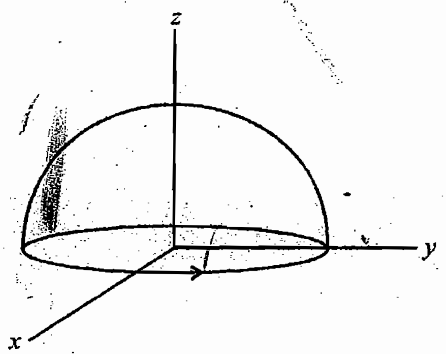


Figure 8.30

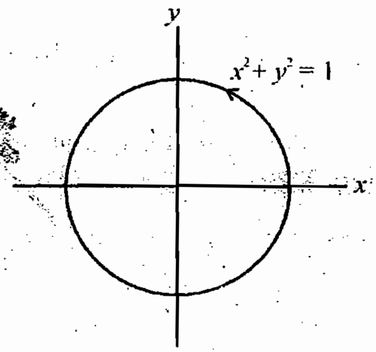


Figure 8.31

$$\bar{F}\cdot d\bar{r} = ydx+(x-2xz)dy-xydz$$
$$= -\sin^2\theta d\theta+\cos^2\theta d\theta = \cos 2\theta d\theta$$

By Stoke's law

$$\int_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \oint \bar{F}\cdot d\bar{r}$$

$$= \int_0^{2\pi}\cos 2\theta d\theta = \frac{1}{2}\sin 2\theta\Big|_0^{2\pi} = 0$$

26. Evaluate the surface integral $\int_S \nabla\times\bar{F}\cdot\hat{n}\,dS$ by transforming it into a line integral, S being that part of the surface of the paraboloid $z=1-x^2-y^2$ for which $z\geq 0$ and $\bar{F}=y\hat{i}+z\hat{j}+x\hat{k}$.

Solution.

S is the surface of paraboloid $z = 1 - x^2 - y^2$ lying above xy plane and bounded by curve C, which is a circle $x^2 + y^2 = 1$ lying in xy plane

$$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$$

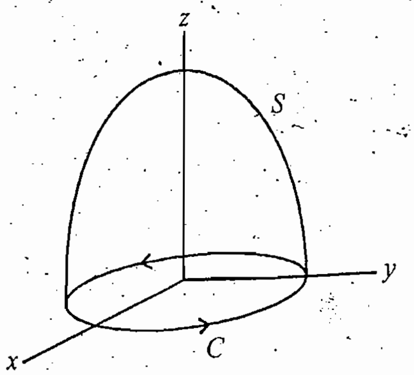


Figure 8.32

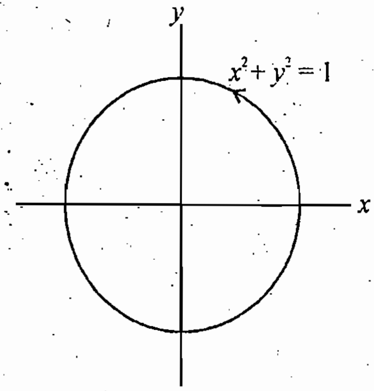


Figure 8.33

$$\vec{F}\cdot d\vec{r} = ydx + zdy + xdz$$

on C,
$$x = \cos\theta \Rightarrow dx = -\sin\theta d\theta$$
$$y = \sin\theta \Rightarrow dy = \cos\theta d\theta$$
$$z = 0 \Rightarrow dz = 0$$

So, $$\vec{F}\cdot d\vec{r} = -\sin^2\theta d\theta$$

So, by Stoke's theorem

$$\int \nabla\times\vec{F}\cdot\hat{n}\,dS = \oint \vec{F}\cdot d\vec{r}$$
$$= -\int_0^{2\pi} \sin^2\theta d\theta$$
$$= -\pi$$

27. If $\vec{F} = (y^2 + z^2 - x^2)\hat{i} + (z^2 + x^2 - y^2)\hat{j} + (x^2 + y^2 - z^2)\hat{k}$, evaluate $\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$ taken over the portion of the surface $x^2 + y^2 + z^2 - 2x + z = 0$ above the plane $z = 0$ and verify Stoke's theorem.

Solution.

S is a part of sphere $x^2 + y^2 + z^2 - 2x + z = 0$ i.e. $(x-1)^2 + y^2 + \left(z + \frac{1}{2}\right)^2 = \frac{5}{4}$ of radius $\frac{\sqrt{5}}{2}$. It is bounded by circle $x^2 + y^2 - 2x = 0$ lying in xy plane.

Let us first evaluate the surface integral $\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$. Consider a closed piecewise surface Σ consisting of hemisphere S and its base in xy plane S'.

By Gauss divergance theorem

$$\oint_\Sigma \nabla\times\vec{F}\cdot\hat{n}\,dS = \int \nabla\cdot(\nabla\times\vec{F})d\tau = 0$$

$$\Rightarrow \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS + \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS = 0$$

So, $$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$

$$\vec{F} = (y^2+z^2-x^2)\hat{i}+(z^2+x^2-y^2)\hat{j}+(x^2+y^2+z^2)\hat{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix}$$

$$= 2(y-z)\hat{i}+2(z-x)\hat{j}+2(x-y)\hat{k}$$

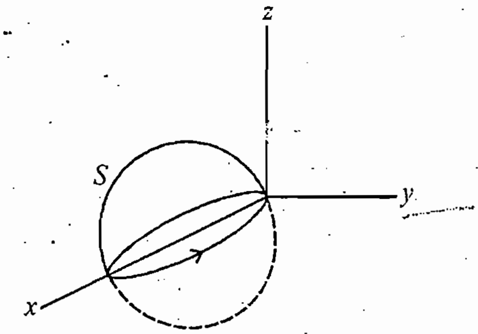


Figure 8.34

On S', $\hat{n}=-\hat{k}$, $\nabla\times\vec{F}\cdot\hat{n}=-2(x-y)$

$$dS = dxdy$$

So,
$$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$

$$= 2\iint (x-y)\,dxdy$$

$$= 2\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r(\cos\theta-\sin\theta)\,r\,dr\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} (\cos\theta-\sin\theta)\left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{16}{3}\int_{-\pi/2}^{\pi/2} (\cos\theta-\sin\theta)\cos^3\theta\, d\theta$$

$$= \frac{16}{3}\left[\int_{-\pi/2}^{\pi/2} (\cos^4\theta d\theta - \int_{-\pi/2}^{\pi/2} \cos^3\theta\sin\theta\, d\theta\right]$$

$$= \frac{32}{3}\int_0^{\pi/2} \cos^4\theta d\theta = \frac{32}{3}\cdot\frac{\overline{|5/2}\,\overline{|1/2}}{2\overline{|3}} = 2\pi$$

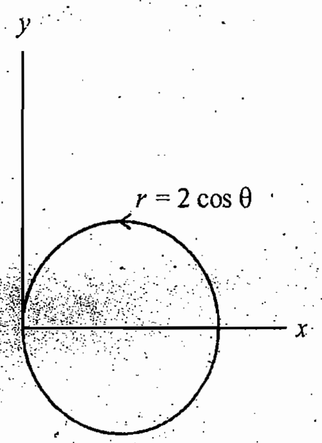


Figure 8.35

Let us now evaluate the line integral $\oint_C \vec{F}\cdot d\vec{r}$.

The curve C is a circle $(x-1)^2+y^2=1$ in xy plane.

$x = 1+\cos\theta,\ dx = -\sin\theta\, d\theta$

$y = \sin\theta,\ dy = \cos\theta\, d\theta$

$z = 0,\ dz = 0$

$$\vec{F}\cdot d\vec{r} = (y^2+z^2-x^2)\,dx + (z^2+x^2-y^2)\,dy + (x^2+y^2-z^2)\,dz$$

$$= (y^2-x^2)\,dx + (x^2-y^2)\,dy$$

$$= (x^2-y^2)\,(dy-dx)$$

$$= (1+\cos^2\theta+2\cos\theta-\sin^2\theta)\cdot(\cos\theta+\sin\theta)\,d\theta$$

$$= (\cos^3\theta-\sin^3\theta+\cos^2\theta\sin\theta-\sin^2\theta\cos\theta+2\cos^2\theta$$
$$+2\cos\theta\sin\theta+\cos\theta+\sin\theta)d\theta$$

So,
$$\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (\cos^3\theta-\sin^3\theta+\cos^2\theta\sin\theta-\sin^2\theta\cos\theta+2\cos^2\theta$$
$$+2\cos\theta\sin\theta+\cos\theta+\sin\theta)d\theta$$

$$= 2\int_0^{2\pi} \cos^2\theta d\theta$$

$$= 2\pi$$

Hence, $\oint \vec{F}\cdot d\vec{r} = \int \nabla\times\vec{F}\cdot\hat{n}\,dS$

So, Stoke's theorem is verified.

28. By converting into a line integral, evaluate $\int \nabla\times\vec{A}\cdot\hat{n}\,dS$ when $\vec{A} = (x-z)\hat{i} + (x^3+yz)\hat{j} - 3xy^2\hat{k}$ and S is the surface of the cone $z = 2-\sqrt{x^2+y^2}$ above the xy plane.

Solution.

S is the surface of cone $z = 2-\sqrt{x^2+y^2}$ above the xy plane. It is bounded by curve C in xy plane.

The curve C: is circle of radius 2 and centre at origin in xy plane.

On curve C, $x = 2\cos\theta, y = 2\sin\theta, z = 0$

$dx = -2\sin\theta d\theta, dy = 2\cos\theta d\theta, dz = 0$

$$\vec{A}\cdot d\vec{r} = (x-z)dx + (x^3+yz)dy - 3xy^2dz$$

$$= xdx + x^3dy \qquad \text{(as } z = 0, dz = 0 \text{ on } C\text{)}$$

$$= -4\cos\theta\sin\theta d\theta + 16\cos^4\theta\cdot d\theta$$

On C, θ varies from 0 to 2π

$$\oint \vec{A}\cdot d\vec{r} = -4\int_0^{2\pi}\cos\theta\sin\theta d\theta + 16\int_0^{2\pi}\cos^4\theta d\theta$$

$$= 0 + 64\int_0^{\pi/2}\cos^4\theta d\theta$$

$$= 64\cdot\frac{\Gamma\frac{5}{2}\Gamma\frac{1}{2}}{2\Gamma 3} = 12\pi$$

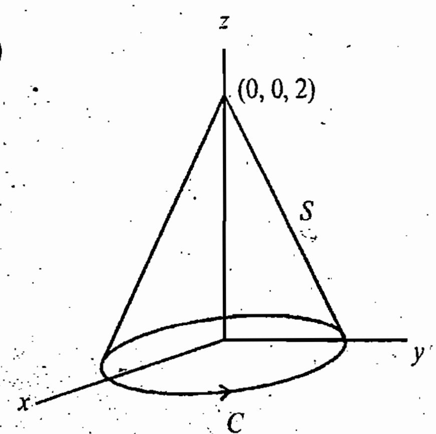


Figure 8.36

29. By converting into a line integral, evaluate $\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$,

where $\vec{F} = (x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^3)\hat{k}$ and S is the surface of

(i) the hemisphere $x^2+y^2+z^2 = a^2$ above xy plane

(ii) the paraboloid $z = 9-(x^2+y^2)$ above the xy plane.

Solution.

(i) S is the surface of hemisphere $x^2+y^2+z^2 = a^2$ above the xy plane and bounded by the circle C $x^2+y^2 = a^2$ in xy plane as shown in Fig.8.37.

On C, $x = a\cos\theta, y = a\sin\theta, z = 0$

$$dx = -a\sin\theta d\theta, dy = a\cos\theta d\theta, dz = 0$$

$$\vec{F}\cdot d\vec{r} = (x^2+y-4)dx + 3xydy + (2xz+z^3)dz$$

$$= (a^2\cos^2\theta + a\sin\theta - 4)\cdot(-a\sin\theta d\theta) + 3a^2\cos\theta\sin\theta(a\cos\theta)d\theta$$

$$= (-a^2\cos^2\theta\sin\theta - a^2\sin^2\theta + 4a\sin\theta + 3a^3\cos^2\theta\sin\theta)d\theta$$

By Stoke's theorem

$$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi}(-a^3\cos^2\theta\sin\theta - a^2\sin^2\theta + 4a\sin\theta$$

$$+ 3a^3\cos^2\theta\sin\theta)\,d\theta$$

$$= -a^2\int_0^{2\pi}\sin^2\theta\,d\theta$$

$$= -\pi a^2$$

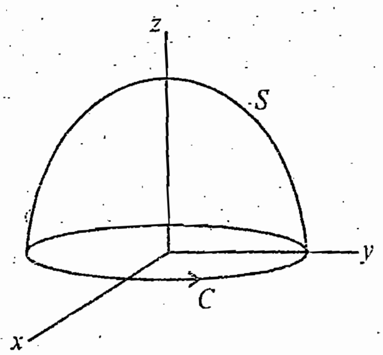


Figure 8.37

(ii) S is the surface of paraboloid above xy plane bounded by curve C in xy plane.
Bounding curve C is circle $x^2+y^2=9$ of radius 3 and centre at origin as shown in fig.8.39
On C, $x=3\cos\theta$, $y=3\sin\theta$, $z=0$
$dx=-3\sin\theta\,d\theta$, $dy=3\cos\theta\,d\theta$, $dz=0$

$$\vec{F}\cdot d\vec{r} = (x^2+y-4)dx + 3xy\,dy + (2xz+z^3)dz$$

$$= (9\cos^2\theta+3\sin\theta-4)\cdot(-3\sin\theta\,d\theta) + 81\cos^2\theta\sin\theta\,d\theta + 0$$

$$= (54\cos^2\theta\sin\theta - 9\sin^2\theta + 12\sin\theta)\,d\theta$$

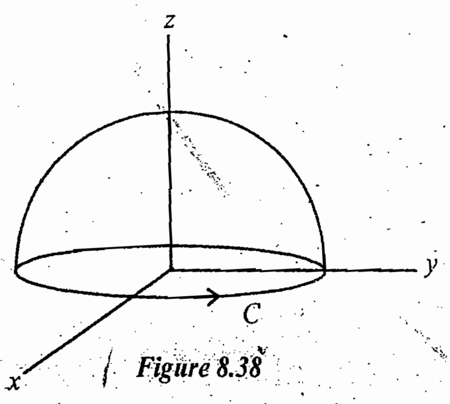


Figure 8.38

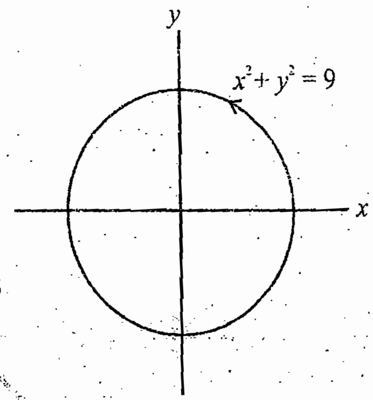


Figure 8.39

So, By Stoke's theorem

$$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi}(54\cos^2\theta\sin\theta - 9\sin^2\theta + 12\sin\theta)\,d\theta$$

$$= -9\int_0^{2\pi}\sin^2\theta\,d\theta$$

$$= -9\pi$$

30. Verify Stoke's for the vector $\vec{F}=3y\,\hat{i}-xz\,\hat{j}+yz^2\hat{k}$ where S is the surface of the paraboloid $z=x^2+y^2$ bounded by $z=4$ and C is its boundary.

Solution.

S is the surface of paraboloid $z=x^2+y^2$ and bounded by $z=4$.
The bounding curve C will be a circle $x^2+y^2=4$, $z=4$.

For a given surface, if $\hat{n}$ is an outward drawn normal, then the corresponding orientation of curve will be clockwise if seen from above.

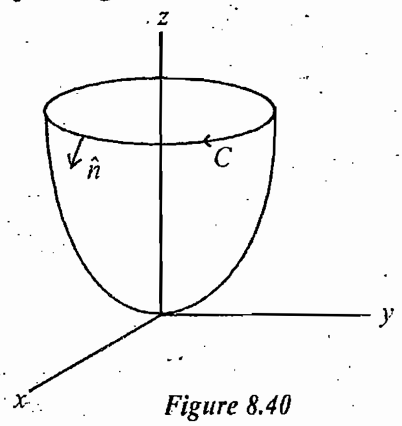


Figure 8.40

Let us first evaluate the line integral

On C, $\quad x = 2\cos\theta,\ y = 2\sin\theta,\ z = 4$

$$dx = -2\sin\theta\, d\theta,\ dy = 2\cos\theta\, d\theta,\ dz = 0$$

$$\vec{F}\cdot d\vec{r} = 3y\,dx - xz\,dy + yz^2 dz$$

$$= -120\sin^2\theta d\theta - 16\cos^2\theta d\theta + 0$$

$$\oint \vec{F}\cdot d\vec{r} = -12\int_{2\pi}^{0}\sin^2\theta d\theta - 16\int_{2\pi}^{0}\cos^2\theta d\theta$$

$$= 28\pi$$

Now, consider a closed piecewise smooth surface Σ consisting of parabolic part S and base S' at $(z = 4)$.

So, by Gauss divergence theorem

$$\oint_{\Sigma} \nabla\times\vec{F}\cdot\hat{n}\,dS = \int \nabla\cdot(\nabla\times\vec{F})d\tau = 0$$

$$\Rightarrow \quad \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS + \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS = 0$$

So, $\quad \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$

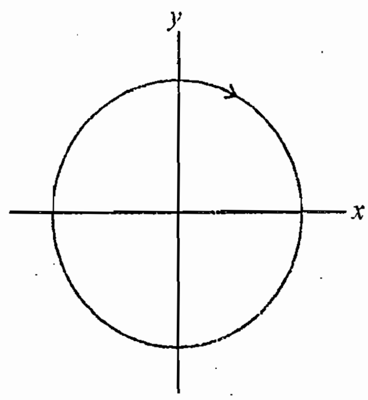


Figure 8.41

On S', $\hat{n} = \hat{k}$, $\therefore\ dS = dxdy$

$$\vec{F} = 3y\hat{i} - xz\hat{j} + yz^2\hat{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix}$$

$$= (z^2 + x)\hat{i} + 0\hat{j} - (z+3)\hat{k}$$

On S', $\quad \nabla\times\vec{F}\cdot\hat{n} = -(z+3) = -7 \quad (z = 4 \text{ on } S)$

So, $\quad \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$

$$= 7\int_{S'} dS$$

$$= 7 \times \text{Area of base}$$

$$= 28\pi$$

Since $\quad \oint \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$

Hence, Stoke's theorem is verified.

31. Use Stoke's theorem to evaluate the line integral $\oint_C x^2y^3dx + dy + zdz$ where C is the circle $x^2 + y^2 = 4,\ z = 0.$

Solution.

The curve C is a circle $x^2 + y^2 = 4$ in xy plane. Let the surface enclosed is a disc of radius 2 lying in xy plane and bounded by C.

For anticlockwise orientation, $\hat{n}=\hat{k}$

$$\oint x^2y^3dx+dy+z\,dz = \oint(x^2y^3\hat{i}+\hat{j}+z\hat{k})\cdot d\vec{r} = \oint \vec{F}\cdot d\vec{r}$$

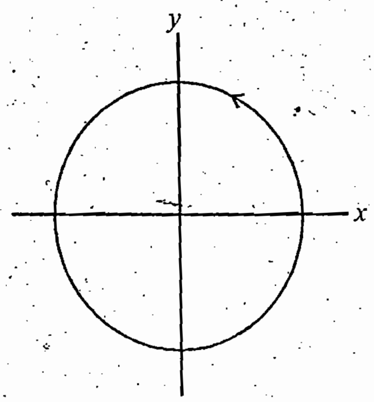


Figure 8.42

So, $\vec{F} = x^2y^3\hat{i}+\hat{j}+z\hat{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = -3x^2y^2\,\hat{k}$$

On S, $\hat{n}=\hat{k}$, $dS = dxdy$, $z=0$

$$\nabla\times\vec{F}\cdot\hat{n} = -3x^2y^2$$

So, by Stoke's theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$$

$$= -3\iint_S x^2y^2\,dxdy$$

$$= -3\int_0^{2\pi}\int_0^2 r^5\sin^2\theta\cos^2\theta\,drd\theta$$

$$= -3\int_0^{2\pi}\left[\frac{r^6}{6}\right]_0^2 \sin^2\theta\cos^2\theta\,d\theta$$

$$= -128\frac{\boxed{3/2}\,\boxed{3/2}}{2\cdot\boxed{3}}$$

$$= -8\pi$$

32. Evaluate $\iint (y-z)\,dydz+(z-x)\,dzdx+(x-y)\,dxdy$ where S is the portion of the surface $x^2+y^2-2x+z=0$, $z\geq 0$.

Solution.

$$\iint (y-z)dydz+(z-x)dzdx+(x-y)dxdy$$

$$= \iint((y-z)\hat{i}+(z-x)\hat{j}+(x-y)\hat{k})\cdot(dydz\,\hat{i}+dzdx\,\hat{j}+dxdy\,\hat{k})$$

$$= \int((y-z)\hat{i}+(z-x)\hat{j}+(x-y)\hat{k})\cdot\hat{n}\,dS = \int(\nabla\times\vec{F})\cdot\hat{n}\,dS$$

So, $\nabla\times\vec{F} = (y-z)\hat{i}+(z-x)\hat{j}+(x-y)\hat{k}$

if $\vec{F} = f\,\hat{i}+g\,\hat{j}+h\hat{k}$

$$\Rightarrow \left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\hat{i}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\hat{j}\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\hat{k}$$

$$= (y-z)\hat{i}+(z-x)\hat{j}+(x-y)\hat{k}$$

$$\frac{\partial f}{\partial y} = y$$

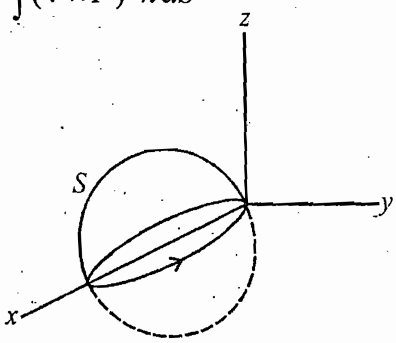


Figure 8.43

$$\Rightarrow \quad f = \frac{y^2}{2}$$

$$\frac{\partial f}{\partial z} = z$$

$$\Rightarrow \quad f = \frac{z^2}{2}$$

So, $$f = \frac{y^2+z^2}{2}$$

Similarly, $$g = \frac{x^2+z^2}{2}$$

$$h = \frac{x^2+y^2}{2}$$

So, $$\vec{F} = \frac{1}{2}(y^2+z^2)\hat{i} + \frac{1}{2}(z^2+x^2)\hat{j} + \frac{1}{2}(x^2+y^2)\hat{k}$$

S is the portion of surface $x^2+y^2-2x+z=0$ above xy plane. The bounding curve of this surface is $x^2+y^2-2x=0$.

i.e. $\quad (x-1)^2+y^2=1$

On C, $x=1+\cos\theta, y=\sin\theta, z=0$

$dx = -\sin\theta\, d\theta,\ dy = \cos\theta\, d\theta$

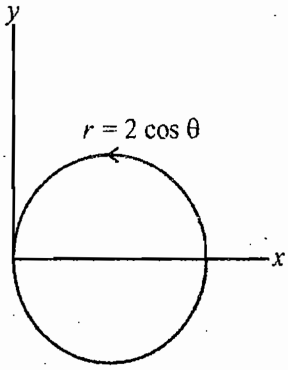


Figure 8.44

$$\vec{F}\cdot d\vec{r} = \frac{1}{2}(y^2+z^2)dx + \frac{1}{2}(z^2+x^2) + \frac{1}{2}(x^2+y^2)dz$$

$$= \frac{1}{2}y^2dx + \frac{1}{2}x^2dy$$

$$= \frac{1}{2}\left[\sin^2\theta(-\sin\theta d\theta) + (1+\cos\theta)^2\cos\theta d\theta\right]$$

$$= \frac{1}{2}(\cos^3\theta + 2\cos^2\theta + \cos\theta - \sin^3\theta)d\theta$$

By Stoke's theorem

$$\int_S \nabla\times\vec{F}\cdot\hat{n}\, dS = \oint \vec{F}\cdot d\vec{r}$$

$$= \frac{1}{2}\int_0^{2\pi}(\cos^3\theta + 2\cos^2\theta + \cos\theta - \sin^3\theta)d\theta$$

$$= \int_0^{2\pi}\cos^2\theta\, d\theta = \pi$$

33. Evaluate using Stoke's theorem

$$\oint_C (y+z)\,dx + (z+x)dy + (x+y)dz$$

where C is the circle $x^2+y^2+z^2=1, x+y+z=0$.

Solution.

The bonding curve C is the curve of intersection of sphere $x^2+y^2+z^2=1$ and plane $x+y+z=0$. Let the surface S be a disc of radius 1 with centre at origin bounded by C

$\oint((xy+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k})\cdot d\vec{r}$

So, $\vec{F} = (y+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = 0$$

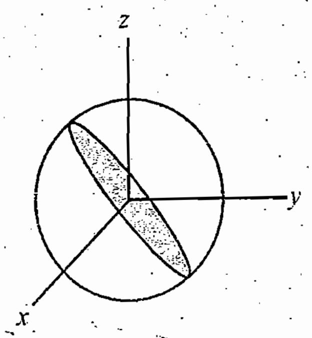


Figure 8.45

By Stoke's theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = 0$$

34. Verify Stoke's theorem for the integral $\oint_C x^2dx+xydy$ where C is the square in the $z=0$ plane with sides along the lines $x=0, y=0, x=1, y=1$.

Solution.

C is a piecewise smooth curve consisting of $C_1: y=0, C_2: x=1, C_3: y=1, C_4: x=0$.
The surface S bounded by C is a square of length of sides 1 unit lying in xy plane.

On C_1, $y=0$, $dy=0$

$$\vec{F}\cdot d\vec{r} = x^2dx + xydy = x^2dx \qquad (y=0)$$

x varies from 0 to 1.

$$\int_{C_1}\vec{F}\cdot d\vec{r} = \int_0^1 x^2dx = \frac{1}{3}$$

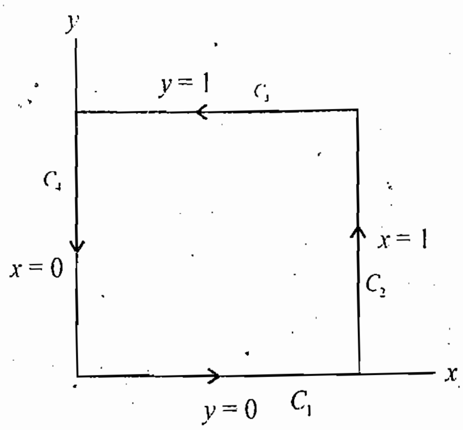


Figure 8.46

On C_2, $x=1$, $dx=0$

$$\vec{F}\cdot d\vec{r} = ydy, \quad y \text{ varies from 0 to 1}$$

$$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_0^1 ydy = \frac{1}{2}$$

On C_3, $y=1$, $dy=0$

$$\vec{F}\cdot d\vec{r} = x^2dx, \quad x \text{ varies from 1 to 0}$$

$$\int_{C_3}\vec{F}\cdot d\vec{r} = \int_1^0 x^2dx = -\frac{1}{3}$$

On C_4, $x=0$, $dx=0$

$$\vec{F}\cdot d\vec{r} = 0$$

$$\int_{C_4}\vec{F}\cdot d\vec{r} = 0$$

Hence,

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r} + \int_{C_4}\vec{F}\cdot d\vec{r}$$

$$= \frac{1}{3}+\frac{1}{2}-\frac{1}{3}+0$$

$$= \frac{1}{2}$$

Now, Let us evaluate the surface integral on S

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= y\hat{k}$$

For given orientation of C, $\hat{n}=\hat{k}$

$$\nabla\times\vec{F}\cdot\hat{n} = y$$

$$\int \nabla\times\vec{F}\cdot\hat{n}\,dS = \int_0^1\int_0^1 y\,dx\,dy$$

$$= \int_0^1 \left.\frac{y^2}{2}\right|_0^1 dx$$

$$= \frac{1}{2}\int_0^1 dx = \frac{1}{2}$$

Hence, $\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$

Thus, Stoke's theorem is verified.

35. Verify Stoke's theorem for $\vec{F} = z\hat{i}+x\hat{j}+y\hat{k}$, S is a part of the paraboloid $z=1-x^2-y^2$ for which $z\geq 0$, $\hat{n}\cdot\hat{k}>0$.

Solution.

The surface S is a part of paraboloid $z=1-x^2-y^2$ lying above xy plane.

Since, $\hat{n}\cdot\hat{k}>0$, $\hat{n}$ is an outward drawn normal to S. For given $\hat{n}$, the curve C is oriented as shown in Fig.

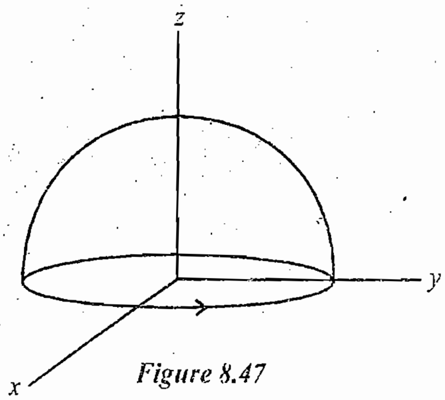


Figure 8.47

C is a circle of radius 1 and centre at origin.

On C, $x=\cos\theta, y=\sin\theta, z=0$

$$dx=-\sin\theta d\theta, dy=\cos\theta d\theta, dz=0$$

$$\vec{F}\cdot d\vec{r} = zdx+xdy+ydz$$

$$= xdy = \cos^2\theta d\theta$$

θ varies from 0 to 2π

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\cos^2\theta d\theta = \pi$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \hat{i}+\hat{j}+\hat{k}$$

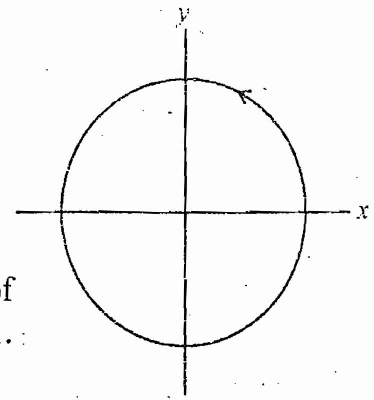


Figure 8.48

Consider a closed piecewise smooth surface Σ consisting of paraboloid $S: z=1-x^2-y^2$ and base of paraboloid $S': x^2+y^2=1$. By Gauss Divergance theorem.

$$\oint_{\Sigma} \nabla\times\vec{F}\cdot\hat{n}\,dS = \int \nabla\cdot(\nabla\times\vec{F})\,d\tau = 0$$

$$\Rightarrow \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS + \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS = 0$$

$$\Rightarrow \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$

On S', $\hat{n} = -\hat{k}$, $dS = dxdy$

$$\nabla\times\vec{F}\cdot\hat{n} = (\hat{i}+\hat{j}+\hat{k})\cdot(-\hat{k}) = -1$$

So, $$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$

$$= \int_{S'} dS = \text{area of base}$$

$$= \pi$$

Since, $$\oint_C \vec{F}\cdot d\vec{r} = \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$

Stoke's theorem is verified.

36. Verify Stoke theorem for $\vec{F} = y^2\hat{i} + xy\,\hat{j} - 2xz\,\hat{k}$ S is the hemisphere $x^2+y^2+z^2=1$, $z \geq 0$ with $\hat{n}\cdot\hat{k} > 0$.

Solution.

The surface S is a hemisphere $x^2+y^2+z^2=1$ lying in xy plane and the bounding curve is $x^2+y^2=1$, $z=0$.

On C, $x = \cos\theta$, $y = \sin\theta$, $z = 0$

$$dx = -\sin\theta\,d\theta,\ dy = \cos\theta\,d\theta,\ dz = 0$$

$$\vec{F} = y^2\,\hat{i} + xy\,\hat{j} - 2xz\,\hat{k}$$

$$\vec{F}\cdot d\vec{r} = y^2dx + xydy - zxzdz$$

$$= -\sin^3\theta d\theta + \sin\theta\cos^2\theta d\theta$$

$$\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (\sin\theta\cos^2\theta - \sin^3\theta)\,d\theta = 0$$

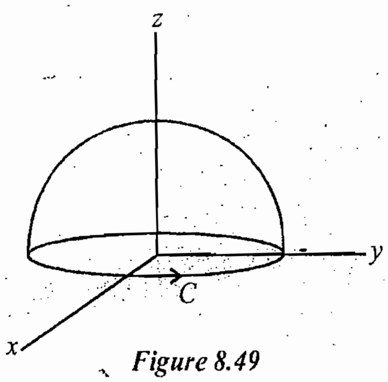


Figure 8.49

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & -2xz \end{vmatrix}$$

$$= 2z\hat{j} - y\hat{k}$$

Consider a closed piecewise smooth surface Σ consisting of hemispherical surfaces S and circular base S'.

Using Gauss divergence theorem

$$\oint_{\Sigma} \nabla\times\vec{F}\cdot\hat{n}\,dS = \int_V \nabla\cdot(\nabla\times\vec{F})\,d\tau = 0$$

$$\Rightarrow \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS + \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS = 0$$

$$\Rightarrow \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\,dS$$

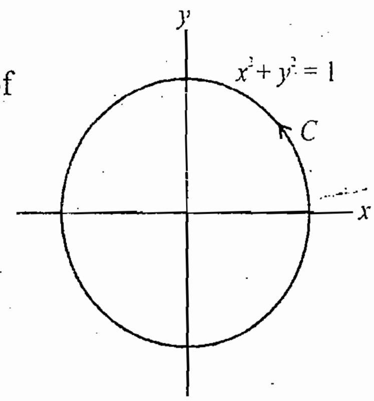


Figure 8.50

On $S', \hat{n} = -\hat{k}$, $dS = dxdy$, $\nabla \times \vec{F} \cdot \hat{n} = y$

$$\int_S \nabla \times \vec{F} \cdot \hat{n}\, dS = \iint y\, dxdy$$

$$= \int_0^1 \int_0^{2\pi} r^2 \sin\theta\, d\theta\, dr = 0$$

Since, $$\oint_C \vec{F} \cdot d\vec{r} = \int_S \nabla \times \vec{F} \cdot \hat{n}\, dS$$

Hence, Stoke's theorem is verified.

37. Verify Stoke's theorem when $\vec{F} = (2xy - y^2)\hat{i} - (x^2 - y^2)\hat{j}$ and C is the boundary of the region enclosed by the parabolas $y^2 = x$ & $x^2 = y$.

Solution.

The curve C is a piecewise smooth curve consisting of $C_1 : x^2 = y$ and $C_2 : y^2 = x$

On C_1, $y = x^2$, $dy = 2xdx$

$$\vec{F} \cdot d\vec{r} = (2xy - y^2)dx - (x^2 - y^2)dy$$
$$= (2x^3 - x^4)dx - (x^2 - x^2)2xdx$$
$$= (2x^5 - x^4)dx$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 (2x^5 - x^4)\, dx$$

$$= \left. \frac{x^6}{3} - \frac{x^5}{5} \right|_0^1$$

$$= \frac{2}{15}$$

On C_2, $x = y^2$, $dx = 2ydy$

$$\vec{F} \cdot d\vec{r} = (2xy - y^2)dx - (x^2 - y^2)dy$$
$$= (2y^3 - y^2) \cdot 2ydy - (y^4 - y^2)dy$$
$$= (3y^4 - 2y^3 + y^2)dy$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_1^0 (3y^4 - 2y^3 + y^2)\, dy$$

$$= \left. \frac{3}{5}y^5 - \frac{y^4}{2} + \frac{y^3}{3} \right|_1^0$$

$$= -\frac{13}{30}$$

So, $$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \frac{2}{15} - \frac{13}{30}$$

$$= -\frac{3}{10}$$

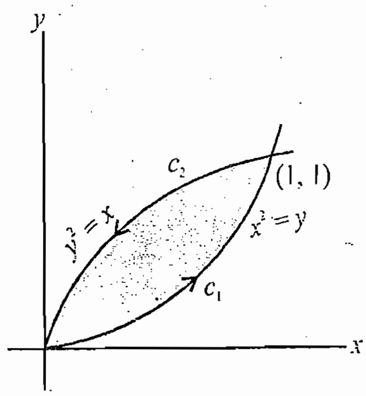


Figure 8.51

Let S be the surface enclosed by curve C.

For the anticlockwise orientation of curve C, the normal to the surface S, $\hat{n} = \hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy - y^2 & y^2 - x^2 & 0 \end{vmatrix}$$

$$= (-4x + 2y)\hat{k}$$

On S, $\nabla \times \vec{F} \cdot \hat{n} = (-4x + 2y)$

$$dS = dxdy$$

$$\int_S \nabla \times \vec{F} \cdot \hat{n}\, dS = \int_0^1 \int_{y^2}^{\sqrt{y}} (-4x + 2y)\, dxdy$$

$$= \int_0^1 -2x^2 + 2xy \Big|_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 \left[-2(y - y^4) + 2y\left(\sqrt{y} - y^2\right)\right] dy$$

$$= \int_0^1 (2y^4 - 2y^3 + 2y^{3/2} - 2y)\, dy$$

$$= \frac{2}{5}y^5 - \frac{1}{2}y^4 + \frac{4}{5}y^{5/2} - y^2 \Big|_0^1 = -\frac{3}{10}$$

Since, $\oint \vec{F} \cdot d\vec{r} = \int_S \nabla \times \vec{F} \cdot \hat{n}\, dS$

Hence, Stoke's theorem is verified.

38. Evaluate $\int_S \text{curl}\, \vec{A} \cdot \hat{n}\, dS$ where S is the open surface $x^2 + y^2 - 4x + 4z = 0$, $z \geq 0$ and $\vec{A} = (y^2 + z^2 - x^2)\hat{i} + (2z^2 + x^2 - y^2)\hat{j}$.

Solution.

S is the open surface $x^2 + y^2 - 4x + 4z = 0$ (paraboloid) above xy plane. The bounding curve C of the surface S is given by $x^2 + y^2 - 4x = 0 \Rightarrow (x-2)^2 + y^2 = 4$ i.e. circle of radius 2 with origin (2, 0) in xy plane.

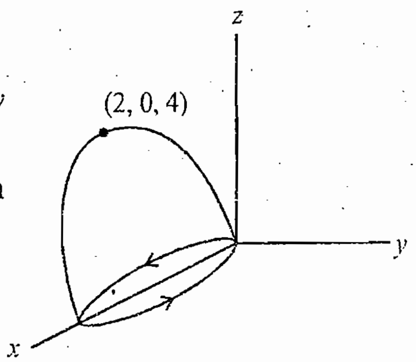


Figure 8.52

On curve C, $x = 2 + 2\cos\theta$, $y = 2\sin\theta$, $z = 0$

$$dx = -2\sin\theta\, d\theta,\ y = 2\cos\theta\, d\theta,\ dz = 0$$

$$\vec{F} \cdot d\vec{r} = (y^2 + z^2 - x^2)dx + (2z^2 + x^2 - y^2)dy$$

$$= (y^2 - x^2)dx + (x^2 - y^2)dy$$

$$= (4\sin^2\theta - 4 - 4\cos^2\theta - 8\cos\theta)(-2\sin\theta)d\theta$$

$$+ (4 + 4\cos^2\theta + 8\cos\theta - 4\sin^2\theta)(2\cos\theta)d\theta$$

$$= 8[-\sin^3\theta+\sin\theta+\cos^2\theta\sin\theta+2\cos\theta\sin\theta$$
$$+\cos\theta+\cos^3+\cos^3\theta+2\cos^2\theta-\sin^2\theta\cos\theta]d\theta$$
$$= 8[\cos^3\theta-\sin^3\theta+\cos^2\theta\sin\theta-\sin^2\theta\cos\theta$$
$$+2\cos^2\theta+2\cos\theta\sin\theta+\cos\theta+\sin\theta]d\theta.$$

θ varies from 0 to 2π

$$\oint \vec{F}\cdot d\vec{r} = 8\int_0^{2\pi}(\cos^3\theta-\sin^3\theta+\cos^2\theta\sin\theta-\sin^2\theta\cos\theta$$
$$+2\cos^2\theta+2\cos\theta\sin\theta+\cos\theta+\sin\theta)d\theta$$
$$= 16\int_0^{2\pi}\cos^2\theta d\theta$$
$$= 16\pi$$

EXERCISE

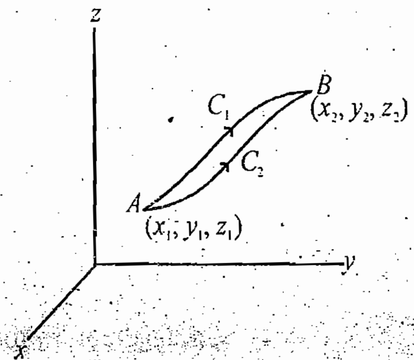
1. Verify Stoke's theorem for $\vec{A}=(y-z+2)\hat{i}+(yz+4)\hat{j}+xz\hat{k}$ where S is the surface of the cube $x=0, y=0, z=0, x=2, y=2, z=2$ above xy plane.
2. Using Stoke's theorem, determine the value of the integral $\int_C ydx+zdy+xdz$ where C is the curve defined by $x^2+y^2+z^2=a^2, x+z=a$. **Ans.** $\sqrt{2}\pi a^2$
3. State Stoke's theorem and the verify it for $\vec{A}=(x^2+1)\hat{i}+xy\hat{j}$ integrated round the square in the plane $z=0$ whose sides are along the lines $x=0, y=0, x=1, y=1$.
4. Verify Stoke's theorem for the vector $\vec{F}=z\hat{i}+x\hat{j}+y\hat{k}$ takes over half of the sphere $x^2+y^2+z^2=a^2$ lying above the xy plane.
5. Verify Stoke's theorem for $\vec{A}=2y\hat{i}+3x\hat{j}+z^2\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=16$ and C is its boundary.
6. Verify Stoke's theorem for the function $\vec{F}=x^2\hat{i}+xy\hat{j}$ intergrated along the rectangle in the plane $z=0$, whose sides are along the lies $x=0, y=0, x=a$ and $y=b$.
7. Verify Stoke's theorem for a vector field defined by $\vec{F}=(x^2-y^2)\hat{i}+2xy\hat{j}$ over the rectangular region in xy plane bounded by lines $x=0, x=a, y=0$ & $y=b$.
8. Verify Stoke's theorem for the function $\vec{F}=x^2\hat{i}+xy\hat{j}$ integrated round the square, in the plane z = 0, whose sides are along the lines $x=0, y=0, x=a, y=a$.
9. Verify Stoke's theorem for the function $\vec{F}=xy\hat{i}+xy^2\hat{j}$ integrated round the square with vertices (1,0,0), (1,1,0), (0,1,0) and (0,0,0).
10. Show that $\oint\phi\vec{F}.d\vec{r}=\int_S\phi\,\text{curl}\vec{F}.\hat{n}dS+\int_S(grad\phi\times\vec{F}).\hat{n}dS$.

9

CONSERVATIVE VECTOR FIELD

CONSERVATIVE VECTOR FIELD

Let $\vec{F}(x,y,z)=f(x,y,z)\hat{i}+g(x,y,z)\hat{j}+h(x,y,z)\hat{k}$ be a vector point function defined and continuous in region R of space. Let A & B are two points in R. Let C_1 and C_2 are arbitrary paths joining A & B. A vector field $\vec{F}(r)$ is said to be conservative if $\int_C \vec{F}(r).d\vec{r}$ from A to B is independent of the paths. In this case the value of line integral depends on initial and final points, A and B and not on the choice of path joining A and B.



For Example, $\vec{F} = yz\hat{i}+xz\hat{j}+xy\hat{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C yzdx + xydy + xydz$$

$$= \int d(xyz) = [xyz]_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}$$

$$x_2y_2z_2 - x_1y_1z_1$$

Till now, we have been incorporating the equation of curve to reduce the line integral into a definite integral involving the independent variable.

If $\vec{F}$ is conservative, the value of integral carried over any arbitrary curve C_1 and C_2 are equal. In case of closed curve, the intial and final points coincide. So, if $\vec{F}(r)$ is conservative vector field, $\oint \vec{F}.d\vec{r}$ over any closed curve will vanish

$$\oint_C \vec{F}.d\vec{r} = 0$$

Using Stoke's theorem

$$\int \nabla\times\vec{F}.\hat{n}dS = 0$$

Since, surface integral of $\int \nabla\times\vec{F}.\hat{n}dS$ is zero over surface enclosed by any arbitrary closed curve.

So, the integrand $\nabla\times\vec{F}$ has to be zero. So, we arrive at alternative condition for conservative vector field.

Theorem : The vector field $\vec{F}(r)$ is said to be conservative iff $\nabla\times\vec{F}=0$.

We also know that curl of gradient is zero.

So, if $\nabla\times\vec{F}=0$.

$\vec{F}$ can be expressed as gradient of a scalar field ϕ.

Theorem. A vector field $\vec{F}(r)$ is said to be conservative iff $\vec{F}$ can be expressed as gradient of a scalar function $\phi(r)$.

So, if $\vec{F}(r)$ is a conservative field

Then $\vec{F}(r) = \nabla\phi(r)$

Where $\phi(r)$ is called potential corresponding to a conservative vector field $\vec{F}(r)$.

For a conservative vector field $\vec{F}(r)$

$$\vec{F}(r)\cdot d\vec{r} = \nabla\phi\cdot d\vec{r}$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$$

$$= d(\phi(x,y,z))$$

So, $\vec{F}(r).d\vec{r}$ is an exact differential.

Theorem : $\vec{F}(r)$ is said to be conservative vector field iff $\vec{F}(r).d\vec{r}$ is an exact differential.

Irrotational field. The vector field $\vec{F}$ is said to be irrotational if $\nabla\times\vec{F} = 0$.

SOLVED EXAMPLES (SOLUTIONS)

1. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = yz\hat{i}+zx\hat{j}+xy\hat{k}$ and C is the portion of the curve $\vec{r} = a\cos t\hat{i} + b\sin t\hat{j} + ct\hat{k}$ from $t = 0$ to $t = \pi/2$.

Solution. $\vec{F}.d\vec{r} = yzdx + zxdy + xydz$

$= d(xyz)$

$\vec{F}.d\vec{r}$ is an exact differential. So, $\vec{F}$ is conservative.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C d(xyz)$$

$$= [xyz]_{t=0}^{t=\pi/2} = [xyz]_{(a,0,0)}^{(0,b,c\pi/2)} = 0$$

Since, $t = 0$ corresponds to point $(0, 0, 0)$ and $t = \pi/2$ corresponds to point $(a, b, \pi/2)$.

2. Evaluate $\int_C (2xy^3 - y^2\cos x)dx + (1-2y\sin x + 3x^2y^2)dy$ where C is the arc of the parabola $2x = \pi y^2$ from $(0, 0)$ to $(\pi/2, 1)$.

Solution.

First let us check whether $(2xy^3 - y^2\cos x)dx + (1-2y\sin x + 3x^2y^2)dy$ is an exact differential.

We know that $Mdx + Ndy$ is an exact differential if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Here, $M = 2xy^3 - y^2\cos x$

$$\frac{\partial M}{\partial y} = 6xy^2 - 2y\cos x$$

$$N = 1 - 2y\sin x + 3x^2y^2$$

$$\frac{\partial N}{\partial x} = -2y\cos x + 6xy^2$$

Thus, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

So, $Mdx + Ndy$ is exact.

Let $Mdx + Ndy$ can be expressed as total differential of a scalar function ϕ

So, $$d\phi = (2xy^3 - y^2 \cos x)dx + (1 - 2y \sin x + 3x^2y^2)dy$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = (2xy^3 - y^2 \cos x)dx + (1 - 2y \sin x + 3x^2y^2)dy$$

$$\frac{\partial \phi}{\partial x} = 2xy^3 - y^2 \cos x$$

$$\Rightarrow \quad \phi = \int_{y=\text{constant}} (2xy^3 - y^2 \cos x)dx = x^2y^3 - y^2 \sin x \qquad \text{....(1)}$$

$$\frac{\partial \phi}{\partial y} = 1 - 2y \sin x + 3x^2y^2$$

$$\Rightarrow \quad \phi = \int_{x=\text{constant}} (1 - 2y \sin x + 3x^2y^2)dy = y - y^2 \sin x + x^2y^3 \qquad \text{....(2)}$$

Adding (1) & (2) while writing the common term once

$$\phi = y - y^2 \sin x - x^2y^3$$

So, given integral

$$\int_C Mdx + Ndy = \int d\phi = \phi$$

$$= [y - y^2 \sin x - x^2 y^3]_{(0,0)}^{(\pi/2,1)} = \frac{\pi^2}{4}$$

3. Show that the given differential form is exact and find a function ϕ such that the form equals $d\phi$

(a) $y^2z^3dx + 2xyz^3dy + 3xy^2z^2dz$ (b) $dx + zdy + ydz$

(c) $\cos x\, dx - 2yzdy - y^2dz$ (d) $(z^2 - 2xy)dx - x^2dy + 2xzdz$

Solution.

(a) $y^2z^3dx + 2xyz^3dy + 3xy^2z^2dz$

$$= (y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}).d\vec{r}$$

Here $$\vec{F} = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyzy)\hat{i} + (3y^2z^2 - 3y^2z^2)\hat{j} + (2yz^3 - 2yz^3)\hat{k} = 0$$

Since, $\nabla \times \vec{F} = 0$, So, $\vec{F}$ can be written as gradient of same scalar function, ϕ.

$$\vec{F} \cdot d\vec{r} = \nabla\phi \cdot d\vec{r} = d\phi$$

Hence, $\vec{F}.d\vec{r}$ is exact differential.

$$F = \nabla\phi$$

So, $$\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}$$

Then, $\frac{\partial\phi}{\partial x} = y^2z^3 \Rightarrow \phi = xy^2z^3 + \text{constant}$ ...(1)

$\frac{\partial\phi}{\partial y} = 2xyz^3 \Rightarrow \phi = xy^2z^3 + \text{constant}$ ...(2)

$\frac{\partial\phi}{\partial z} = 3xy^2z^2 \Rightarrow \phi = xy^2z^3 + \text{constant}$ ...(3)

Adding (1), (2) & (3) writing common term once

$$\phi = xy^2z^3 + c$$

(b) $dx + zdy + ydz = (\hat{i} + z\hat{j} + y\hat{k}).d\vec{r}$

Here $\vec{F} = \hat{i} + z\hat{j} + y\hat{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & z & y \end{vmatrix} = 0$$

Since, $\nabla\times\vec{F} = 0$, $\vec{F}$ can be written as gradient of same scalar function ϕ.

So, $\vec{F}\cdot d\vec{r} = \nabla\phi\cdot d\vec{r} = d\phi$

Hence, $\vec{F}.d\vec{r}$ is an exact differential.

$$\vec{F} = \nabla\phi$$

$$\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \hat{i} + z\hat{j} + y\hat{k}$$

$\frac{\partial\phi}{\partial x} = 1 \Rightarrow \phi = x + \text{constant}$ ...(1)

$\frac{\partial\phi}{\partial y} = z \Rightarrow \phi = yz + \text{constant}$ ...(2)

$\frac{\partial\phi}{\partial z} = y \Rightarrow \phi = yz + \text{constant}$ ...(3)

Adding (1), (2) & (3) and writing the common term once.

$$\phi = x + yz + \text{constant}$$

(c) $\cos x dx - 2yzdy - y^2dz = (\cos x\hat{i} - 2yz\hat{j} - y^2\hat{k}).d\vec{r}$

Here $\vec{F} = \cos x\hat{i} - 2yz\hat{j} - y^2\hat{k}$

Since, $\nabla\times\vec{F} = 0$, so, $\vec{F}$ can be expressed as gradient of scalar function ϕ.

$$\vec{F}\cdot d\vec{r} = \nabla\phi\cdot d\vec{r} = d\phi$$

$$\vec{F} = \nabla\phi$$

$$\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \cos x\hat{i} - 2yz\hat{j} - y^2\hat{k}$$

$\frac{\partial\phi}{\partial x} = \cos x \Rightarrow \phi = \sin x + \text{constant}$ ...(1)

$\frac{\partial\phi}{\partial y} = -2yz \Rightarrow \phi = -y^2z + \text{constant}$ ...(2)

$$\frac{\partial\phi}{\partial z} = -y^2 \quad \Rightarrow \quad \phi = -y^2z + \text{constant} \quad ...(3)$$

Adding (1), (2) & (3) and writing the common terms once

$$\phi = \sin x - y^2z + \text{constant}$$

(d) $(z^2-2xy)dx - x^2dy + 2xzdz = [(z^2-2xy)\hat{i} - x^2\hat{j} + 2xz\hat{k}].d\vec{r}$

Here $\quad \vec{F} = (z^2-2xy)\hat{i} - x^2\hat{j} + 2xz\hat{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2-2xy & -x^2 & 2xz \end{vmatrix} = 0$$

Since, curl $\vec{F} = 0$, $\vec{F}$ can be expressed as gradient of a scalar function ϕ.

So, $\quad \vec{F}\cdot d\vec{r} = \nabla\phi\cdot d\vec{r} = d\phi$

$$\vec{F} = \nabla\phi$$

$$\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = (z^2-2xy)\hat{i} - x^2\hat{j} + 2xz\hat{k}$$

So, $\quad \frac{\partial\phi}{\partial x} = z^2 - 2xy \quad \Rightarrow \quad \phi = z^2x - x^2y + \text{constant} \quad .....(1)$

$$\frac{\partial\phi}{\partial y} = -x^2 \quad \Rightarrow \quad \phi = -x^2y + \text{constant} \quad .....(2)$$

$$\frac{\partial\phi}{\partial z} = 2xz \quad \Rightarrow \quad \phi = xz^2 + \text{constant} \quad .....(3)$$

Adding (1), (2) & (3) and writing the common terms once $\phi = z^2x - x^2y$

4. Evaluate $\int_C 2xyz^2dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$ where C is any path from (0, 0, 1) to $(1, \pi/4, 2)$.

Solution.

$2xyz^2dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$

$$= [2xyz^2\hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)\hat{k}].d\vec{r}$$

Here, $\quad \vec{F} = 2xyz^2\hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)\hat{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + yz\sin yz)\hat{i}$$
$$+ (4xyz - 4xyz)\hat{j} + (2xz^2 - 2xz^2)\hat{k} = 0$$

Since, curl $\vec{F} = 0$. So, $\vec{F}$ is conservative vector field.
So, the given line integral is independent of path.

Let us find the potential ϕ corresponding to $\vec{F}$.

$$\vec{F} = \nabla\phi$$

$$\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k} = 2xyz^2\hat{i}+(x^2z^2+z\cos yz)\hat{j}+(2x^2yz+y\cos yz)\hat{k}$$

So, $\frac{\partial\phi}{\partial x} = 2xyz^2 \Rightarrow \phi = x^2yz^2 - \text{constant}$ ....(1)

$\frac{\partial\phi}{\partial y} = x^2z^2 + z\cos yz \Rightarrow \phi = x^2yz^2 + \sin yz + \text{constant}$....(2)

$\frac{\partial\phi}{\partial z} = 2x^2yz + y\cos yz \Rightarrow \phi = x^2yz^2 + \sin yz + \text{constant}$....(3)

Adding (1), (2), (3) and adding the common terms once

$$\phi = x^2yz^2 + \sin yz + \text{constant}$$

So, the given line integral

$$\int 2xyz^2dx+(x^2z^2+z\cos yz)dy+(2x^2yz+y\cos yz)dz$$

$$= \int d(x^2yz^2+\sin yz) = [x^2yz^2+\sin yz]_{(0,0,0)}^{(1,\pi/4,2)}$$

$$= \pi + 1$$

5. Evaluate

$$\int_C yzdx+(xz+1)dy+xydz$$

where C is any path from (1, 0, 0) to (2, 1, 4).

Solution.

Here, $yzdx + (xz+1)dy + xydz = (yz\hat{i}+(xz+1)\hat{j}+xy\hat{k}).d\vec{r}$

So, $\vec{F} = yz\hat{i}+(xz+1)\hat{j}+xy\hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz+1 & xy \end{vmatrix} = 0$$

Since, curl $\vec{F} = 0$, So, $\vec{F}$ is a conservative vector field and hence, given line integral is independent of path.

Let us find the potential ϕ corresponding to $\vec{F}$.

$$\vec{F} = \nabla\phi$$

$$\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k} = yz\hat{i}+(xz+1)\hat{j}+xy\hat{k}$$

$\frac{\partial\phi}{\partial x} = yz \Rightarrow \phi = xyz + \text{constant}$ ....(1)

$\frac{\partial\phi}{\partial y} = xz + 1 \Rightarrow \phi = xyz + y + \text{constant}$ ....(2)

$\frac{\partial\phi}{\partial z} = xy \Rightarrow \phi = xyz + \text{constant}$ ....(3)

Adding (1), (2) & (3) and writing the common term once

$$\varphi = xyz + y$$

The given line integral is therefore

$$\int_C yzdx + (xz+1)dy + xydz = \int_{(1,0,0)}^{(2,1,4)} d(xyz+y)$$

$$= [xyz+y]_{(1,0,0)}^{(2,1,4)}$$

$$= 9$$

6. Show that the vector field

$\vec{F} = (2xy - y^4 + 3)\hat{i} + (x^2 - 4xy^3)\hat{j}$

is conservative. Find its potential and also the work done in moving a particle from (1, 0) to (2, 1) along some curve.

Solution.

Vector field $\vec{F}$ is conservative if $\oint \vec{F} \cdot d\vec{r}$ around any closed curve is always zero

By stokes theorem,

$$\oint \vec{F} \cdot d\vec{r} = \int \nabla \times \vec{F} \cdot \hat{n} dS = 0$$

So, for conservative field

$$\nabla \times \vec{F} = 0$$

$$\vec{F} = (2xy - y^4 + 3)\hat{i} + (x^2 - 4xy^3)\hat{j}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - y^4 + 3 & x^2 - 4xy^3 & 0 \end{vmatrix}$$

$$= \left[(2x - 4y^3) - (2x - 4y^3)\right]\hat{k} = 0$$

$$\vec{F} = \nabla\phi$$

$$\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} = (2xy - y^4 + 3)\hat{i} + (x^2 - 4xy^3)\hat{j}$$

$$\frac{\partial\phi}{\partial x} = 2xy - y^4 + 3$$

$$\Rightarrow \quad \phi = \int_{y=\text{const}} (2xy - y^4 + 3)dx$$

$$= x^2y - xy^4 + 3x$$

$$\frac{\partial\phi}{\partial y} = x^2 - 4xy^3$$

$$\Rightarrow \quad \phi = \int_{x=\text{const}} (x^2 - 4xy^3)dy = x^2y - xy^4$$

So, the potential ϕ is given by

$$\phi = x^2y - xy^4 + 3x$$

$$\vec{F} \cdot d\vec{r} = \nabla\phi \cdot d\vec{r} = d\phi$$

$$\int \vec{F} \cdot d\vec{r} = \int d\phi = [\phi]_{(1,0)}^{(2,1)}$$

$$= \left[x^2y - xy^4 + 3x\right]_{(1,0)}^{(2,1)} = 5$$

So, work done in moving a particle from (1, 0) to (2, 1) is equal to 5.

7. If $\vec{F} = \cos y\hat{i} - x\sin y\hat{j}$. Evaluate $\int_C \vec{F}.d\vec{r}$ where C is the curve in the xy plane from $(2, 0)$ to $(0, 2)$.

Solution.

$$\vec{F} = \cos y\hat{i} - x\sin y\hat{j}$$

$$\vec{F}\cdot d\vec{r} = \cos y dx - x \sin y dy$$

Let us check whether $\vec{F}\cdot d\vec{r}$ is exact.

$M = \cos y \quad \Rightarrow \quad \dfrac{\partial M}{\partial y} = -\sin y$

$N = -x\sin y \quad \Rightarrow \quad \dfrac{\partial N}{\partial x} = -\sin y$

Since, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\vec{F}.d\vec{r}$ is an exact and hence, $\vec{F}$ is a conservative vector field.
So, the given line integral is independent of path.

Let us find the potential corresponding to $\vec{F}$.

$$\vec{F} = \nabla\phi$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} = \cos y\hat{i} - x\sin y\hat{j}$$

$$\frac{\partial\phi}{\partial x} = \cos y \quad \Rightarrow \quad \phi = x\cos y + \text{constant}$$

$$\frac{\partial\phi}{\partial y} = -x\sin y \quad \Rightarrow \quad \phi = x\cos y + \text{constant}$$

So, $\phi = x\cos y$

So,
$$\int \vec{F}\cdot d\vec{r} = \int \cos y dx - x\sin y dy$$
$$= \int d(x\cos y)$$
$$= [x\cos y]_{(2,0)}^{(0,2)}$$
$$= -2$$

8. Evaluate

$$\int_C (3y^2 + 2z^2)dx + (6x - 10z)ydy + (4xz - 5y^2)dz$$

along the portion from $(1, 0, 1)$ to $(3, 4, 5)$ of the curve C, which is the intersection of the surfaces $z^2 = x^2 + y^2$ and $z = y + 1$

Solution.

$$\vec{F}\cdot d\vec{r} = d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\frac{\partial\phi}{\partial x} = 3y^2 + 2z^2 \quad \Rightarrow \quad \phi = 3xy^2 + 2xz^2 + \text{constant}$$

$$\frac{\partial\phi}{\partial y} = 6xy - 10yz \quad \Rightarrow \quad \phi = 3xy^2 - 5y^2z + \text{constant}$$

$$\frac{\partial\phi}{\partial z} = 4xz - 5y^2 \quad \Rightarrow \quad \phi = 2xz^2 - 5y^2z + \text{constant}$$

So, $\phi = 3xy^2 + 2xz^2 - 5y^2z$

The line integral $\int_C \vec{F}.d\vec{r}$ along the portion (1, 0, 1) to (3, 4, 5)

$$= \left[3xy^2 + 2xz^2 - 5y^2z\right]_{(1,0,1)}^{(3,4,5)}$$
$$= -108$$

9. Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3z^2x\hat{k}$ is a conservative field. Find its scalar potential and also the work done in moving a particle from (1, –2, 1) to (3, 1, 4).

Solution.

$$\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3z^2x\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3z^2x \end{vmatrix} = 0$$

So, $\vec{F}$ is conservative field, it can be expressed as gradient of scalar ϕ.

$$\vec{F} = \nabla\phi$$

$$\Rightarrow \quad (2xy + z^3)\hat{i} + x^2\hat{j} + 3z^2x\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\frac{\partial\phi}{\partial x} = 2xy + z^3 \quad \Rightarrow \quad \phi = x^2y + xz^3 \quad \text{....(1)}$$

$$\frac{\partial\phi}{\partial y} = x^2 \quad \Rightarrow \quad \phi = x^2y \quad \text{....(2)}$$

$$\frac{\partial\phi}{\partial z} = 3z^2x \quad \Rightarrow \quad \phi = xz^3 \quad \text{....(3)}$$

Adding (1), (2), (3) and writing the common term once

$$\phi = x^2y + xz^3$$

Work done in moving a particle from (1, –2, 1) to (3, 1, 4)

$$W = \int \vec{F}\cdot d\vec{r} = \int d\phi$$

$$= [x^2y + xz^3]_{(1,-2,1)}^{(3,1,4)} = 200$$

10. Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find a scalar potential ϕ such that $\vec{A} = \text{grad}\,\phi$.

Solution.

$$\text{curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} = 0$$

Since, curl $\vec{A} = 0$, so, $\vec{A}$ is irrotational.

$\vec{A}$ can be expressed as gradient of scalar function ϕ.

$$\vec{A} = \nabla\phi$$

$$(6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\frac{\partial\phi}{\partial x} = 6xy + z^3 \quad \Rightarrow \quad \phi = 3x^2y + z^3x \qquad ....(1)$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \quad \Rightarrow \quad \phi = 3x^2y - yz \qquad ....(2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \quad \Rightarrow \quad \phi = xz^3 - yz \qquad ....(3)$$

Adding (1), (2), (3) & writing the common terms once

$$\phi = 3xz^3 - yz + xz^3$$

EXERCISE

1. Evaluate $\oint_C \vec{F}.d\vec{r}$ for the field $\vec{F} = \text{grad}(xy^2z^3)$ where C is the ellipse in which the plane $z = 2x + 3y$ cuts the cylinder $x^2 + y^2 = 12$ counter clockwise as viewed from the positive end of the z–axis looking towards the origin. **Ans.** 0
2. Show that the vector field defined by the vector function $\vec{V} = xyz(yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k})$ is conservative. Find the scalar potential such that $\vec{V} = \text{grad}\,\phi$. Find the work done in moving a particle from (1, 1, 2) to (3, 2, 4).
3. Show that the vector field defined by $\vec{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is irrotational. Find also the scalar u such that $\vec{F} = \text{grad}\,u$.

10

CURVILINEAR CORDINATES

CURVILINEAR COORDINATES

In the cartesian coordinates system, we have an origin and a set of three orthogonal axes passing through it. Any point P is described by three coodinates (x, y, z). To point P, we can associate a unique set of coordinates (u_1, u_2, u_3) called curvilinear coordinates of P, such that (x, y, z) can be expressed is terms of (u_1, u_2, u_3), as

$$x = x(u_1, u_2, u_3),\ y = y(u_1, u_2, u_3),\ z = z(u_1, u_2, u_3) \quad ....(1)$$

(1) can be solved for u_1, u_2, u_3 in terms for x, y, z

$$u_1 = u_1(x, y, z),\ u_2 = u_2(x, y, z),\ u_3 = u_3(x, y, z) \quad ....(2)$$

Equation (1) is transformation equation from curvilinear to cartesian. Equation (2) is transformation equation from cartesian to curvilinear.

For Example: We denote a point P by (r, θ, ϕ) is spherical polar coordinates and by (x,y,z) is cartesian coordinate system

The transformation equation from spherical polar to cartesian coordinates are

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad .... (3)$$

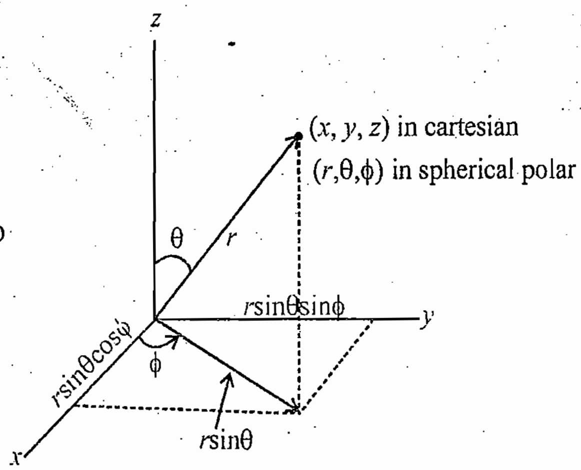


Figure 10.1

The transformation equation from castesian to spherical polar coordinates are

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\sqrt{\frac{x^2 + y^2}{z}}$$

$$\phi = \tan^{-1}\sqrt{\frac{y}{x}}$$

These are transformation equation from castesian to spherical polar.

Note: Unlike $\hat{i}, \hat{j}, \hat{k}$ in cartesian coordinate system, unit vector in direction of r, θ, ϕ i.e. $\hat{e}_r, \hat{e}_\theta, \& \hat{e}_\phi$ are associated with a point P and they change direction as P moves around.

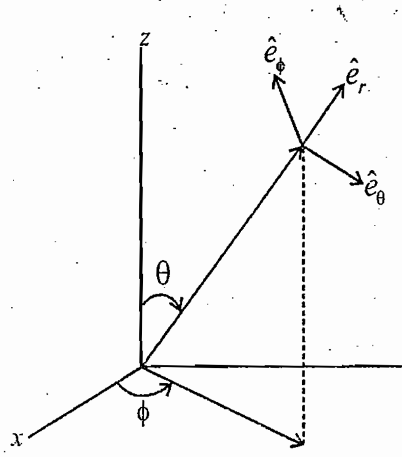


Figure 10.2

ORTHOGONAL CURVILINEAR COORDINATES

Let us discuss about curvilinear coordinates by taking example of spherical polar coordinates.

In spherical polar coordinates, a point P is specified by coordinates (r,θ,ϕ).

The surface $r=c_1$, $\theta=c_2$, $\phi=c_3$ where c_1, c_2, c_3 are constants are called coordinate surface and each pair of these surfaces intersect in a curves called coordinate curves or lines.

Here $r=c_1$ is a sphere of radius r, $\theta=c_2$ is a cone with semivertical angle θ and ϕ is cylinder of radius $r\sin\theta$.

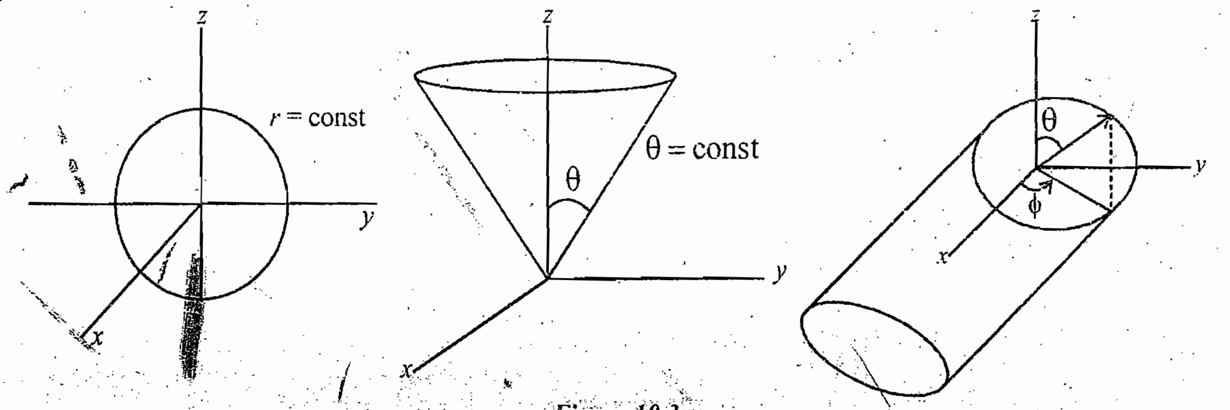


Figure 10.3

In general $u_1=c_1$, $u_2=c_2$, $u_3=c_3$, where c_1, c_2, c_3 are constant, are called coordinate surfaces and each pair of these surfaces intersects in a curve called coordinate curve or lines. If the coordinate surfaces intersects at right angles, the curvilinear coordinate system is called orthogonal. The u_1, u_2, u_3, coordinate curves of a curvilinear system are analogous to the x, y, and z, coordinate axes of a rectangular system.

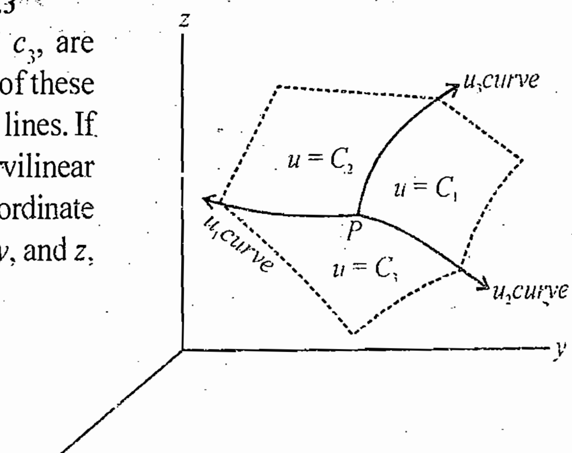


Figure 10.4

Let $\vec{r}$ be the position vector P, $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

x, y, z coordinates can be written in terms of u_1, u_2, u_3 then $\vec{r}$ becomes a vector function of u_1, u_2 & u_3.

For example:

In spherical polar coordinates.

$$\vec{r}=r\sin\theta\cos\phi\hat{i}+r\sin\theta\sin\phi\,\hat{j}+r\cos\theta\hat{k}$$

A tangent vector to the u_1 curve at P (for which u_3 & u_3 are constant) is $\dfrac{\partial\vec{r}}{\partial u_1}$.

Then, unit tangent vector in the direction of tangent to u_1 curve is $\hat{e}_1 = \dfrac{(\partial\bar{r}/\partial u_1)}{|\partial\bar{r}/\partial u_1|}$

Let $$\left|\frac{\partial\bar{r}}{\partial u_1}\right| = h_1$$

So, $$\frac{\partial\bar{r}}{\partial u_1} = h_1\hat{e}_1$$

Similarly, $$\frac{\partial\bar{r}}{\partial u_2} = h_2\hat{e}_2$$

$$\frac{\partial\bar{r}}{\partial u_3} = h_3\hat{e}_3$$

The quantities h_1, h_2, h_3 are called scale factors. The unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are in direction of increasing u_1, u_2, u_3 respectively.

For spherical polar coordinates

$$\bar{r} = r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\hat{j} + r\cos\theta\hat{k}$$

$$\frac{\partial\bar{r}}{\partial r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$h_1 = \left|\frac{\partial\bar{r}}{\partial r}\right| = 1$$

$$\frac{\partial\bar{r}}{\partial\theta} = r\cos\theta\cos\phi\hat{i} + r\cos\theta\sin\phi\hat{j} - r\sin\theta\hat{k}$$

$$h_2 = \left|\frac{\partial\bar{r}}{\partial\theta}\right| = r$$

$$\frac{\partial\bar{r}}{\partial\phi} = -r\sin\theta\sin\phi\hat{i} + r\sin\theta\cos\phi\hat{j}$$

$$h_3 = \left|\frac{\partial\bar{r}}{\partial\phi}\right| = r\sin\theta$$

So, scale factors for spherical polar coordinates are $h_1 = 1, h_2 = r, h_3 = r\sin\theta$

In cylindrical polar coordinates, a point P is defined by coordinates (ρ, ϕ, z)

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= \rho\cos\phi\hat{i} + \rho\sin\phi\hat{j} + z\hat{k}$$

$$\frac{\partial\bar{r}}{\partial\rho} = \cos\phi\hat{i} + \sin\phi\hat{j}$$

$$h_1 = \left|\frac{\partial\bar{r}}{\partial\rho}\right| = 1$$

$$\frac{\partial\bar{r}}{\partial\phi} = \rho\sin\phi\hat{i} + \rho\cos\phi\hat{j}$$

$$h_2 = \left|\frac{\partial\bar{r}}{\partial\phi}\right| = \rho$$

$$\frac{\partial\bar{r}}{\partial z} = \hat{k}$$

$$h_3 = \left|\frac{\partial\bar{r}}{\partial z}\right| = 1$$

So, the scale factors for cylindrical polar coordinates are

$$h_1 = 1$$
$$h_2 = \rho$$
$$h_3 = 1$$

GRADIENT

Let $\bar{r}$ be a vector denoting a point P in space. Let us consider a point Q is the vicinity of point P, described by vector $\bar{r} + d\bar{r}$, $\bar{r}$ is a function of u_1, u_2, u_3.

So, $d\bar{r}$ is also function of u_1, u_2 & u_3

$$d\bar{r} = \frac{\partial \bar{r}}{\partial u_1} du_1 + \frac{\partial \bar{r}}{\partial u_2} du_2 + \frac{\partial \bar{r}}{\partial u_3} du_3$$
$$= h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

Let ϕ be a scalar function which is a function curvilinear coordinates u_1, u_2, u_3.

The differential $d\phi$ can be written as

$$d\phi = \frac{\partial \phi}{\partial u_1} du_1 + \frac{\partial \phi}{\partial u_2} du_2 + \frac{\partial \phi}{\partial u_3} du_3$$

$d\phi$ can also be written as

$$d\phi = \nabla\phi . d\bar{r} \qquad \text{where } \nabla\phi \text{ is the gradient of } \phi$$
$$= \left((\nabla\phi)_{u_1}\hat{e}_1 + (\nabla\phi)_{u_2}\hat{e}_2 + (\nabla\phi)_{u_3}\hat{e}_3\right).\left(h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_2 du_3 \hat{e}_3\right)$$
$$= (\nabla\phi)_{u_1} h_1 du_1 + (\nabla\phi)_{u_2} h_2 du_2 + (\nabla\phi)_{u_3} h_3 du_3$$

Where $(\nabla\phi)_{u_1}$, $(\nabla\phi)_{u_2}$ & $(\nabla\phi)_{u_3}$ are components of $\nabla\phi$ along u_1, u_2, u_3 curves

Comparing both the expression of $d\phi$

$$(\nabla\phi)_{u_1} = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1}$$
$$(\nabla\phi)_{u_2} = \frac{1}{h_2}\frac{\partial \phi}{\partial u_2}$$
$$(\nabla\phi)_{u_3} = \frac{1}{h_3}\frac{\partial \phi}{\partial u}$$

So,
$$\nabla\phi = (\nabla\phi)_{u_1} e_1 + (\nabla\phi)_{u_2}\hat{e}_2 + (\nabla\phi)_{u_3}\hat{e}_3$$
$$= \frac{1}{h_1}\frac{\partial \phi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial u_3}\hat{e}_3$$

u_1, u_2 & u_3 are curvilinear coordinates, h_1, h_2, h_3 are scale factors and $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ are unit vectors along coordinate curves.

The scale factors for different coordinate systems are

System	u_1	u_2	u_3	h_1	h_2	h_3
Cartesian	x	y	z	1	1	1
Spherical polar	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical polar	ρ	ϕ	z	1	ρ	1

DIVERGENCE

Suppose we have a vector function

$$\vec{A}(u_1, u_2, u_3) = A_{u_1}\hat{e}_1 + A_{u_2}\hat{e}_2 + A_{u_3}\hat{e}_3$$

Let us evaluate $\oint \vec{A}\cdot\hat{n}\,dS$ over the surface of the infinitesimal volume beginning at (u_1, u_2, u_3) and increasing each of the coordinates in succession by an infinitesimal amount. Since, the coordinates are orthogonal, it's a cuboid with length of sides $h_1du_1, h_2du_2, h_3du_3$.

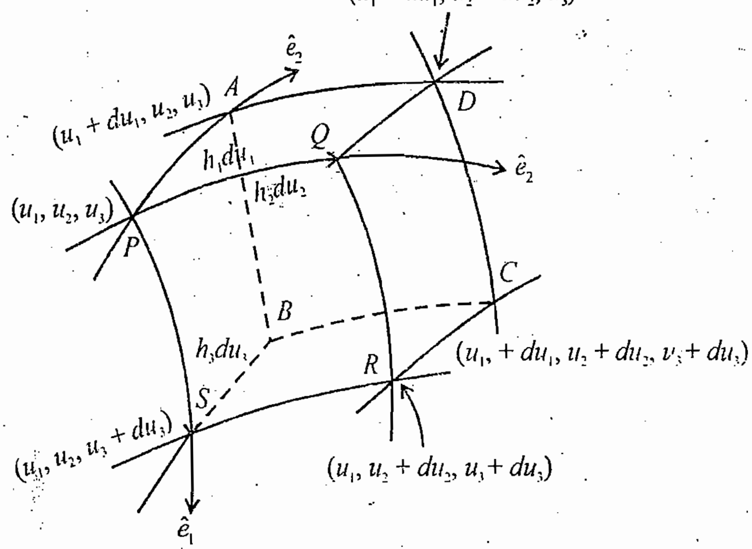


Figure 10.5

The volume of this region is

$$d\tau = h_1h_2h_3du_1du_2du_3$$

For the surface, *PQRS*

$$dS = h_2h_3du_2du_3$$

Outward drawn normal to $dS = -\hat{e}_1$

$$\hat{n}\,dS = -h_2h_3du_2du_3\hat{e}_1$$

$$\vec{A}\cdot\hat{n}\,dS = -\vec{A}\cdot\hat{e}_1h_2h_3du_2du_3$$

$$= -A_{u_1}h_2h_3du_2du_3$$

To evaluate surface integral on back surface *ABCD*, the quantity $A_{u_1}h_2h_3$ has to be evaluated at $u_1 + du_1$

For a derivable function $f(u)$

$$f(u+du) = f(u) + \frac{df}{du}du$$

On *ABCD*, $\hat{n} = \hat{e}_1$

$$dS = du_2du_3$$

$$\vec{A}\cdot\hat{n}\,dS = \vec{A}\cdot\hat{e}_1\,du_2du_3$$

$$= \left(A_{u_1}h_2h_3 + \frac{\partial}{\partial u_1}(h_2h_3A_{u_1})du_2du_3\right)$$

The surface integral across *PQRS* & *ABCD* together gives

$$\frac{\partial}{\partial u_1}(h_2h_3A_{u_1})du_1du_2du_3 = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(h_2h_3A_{u_1})d\tau$$

Similarly, surface integral across *PABS* & *QDCR* together gives

$$\frac{\partial}{\partial u_2}(h_1h_3A_{u_2})du_1du_2du_3 = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_2}(h_1h_3A_{u_2})d\tau$$

The surface integral across *PADQ* & *SBCR* together gives

$$\frac{\partial}{\partial u_3}(h_1h_2A_{u_3})du_1du_2du_3 = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_3}(h_1h_2A_{u_3})d\tau$$

Hence, from across the whole surface

$$\oint \vec{A}\cdot\hat{n}\,dS = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_{u_1})+\frac{\partial}{\partial u_2}(h_3h_1A_{u_2})+\frac{\partial}{\partial u_3}(h_1h_2A_{u_3})\right]d\tau$$

So, from Gauss Divergence theorem

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_{u_1})+\frac{\partial}{\partial u_2}(h_3h_1A_{u_2})+\frac{\partial}{\partial u_3}(h_1h_2A_{u_3})\right]$$

CURL

Suppose we have a vector function

$$\vec{A}(u_1,u_2,u_3) = A_{u_1}\hat{e}_1 + A_{u_2}\hat{e}_2 + A_{u_3}\hat{e}_3$$

Let us evaluate $\oint \vec{A}\cdot\vec{dl}$ around a infinitesimal loop generated by starting at (u_1, u_2, u_3) and successively increasing u_1 and u_2 by infinitesimal amount, holding u_3 constant. The surface bounded by loop is a rectangle of lengths h_1du_1 & h_2du_2.

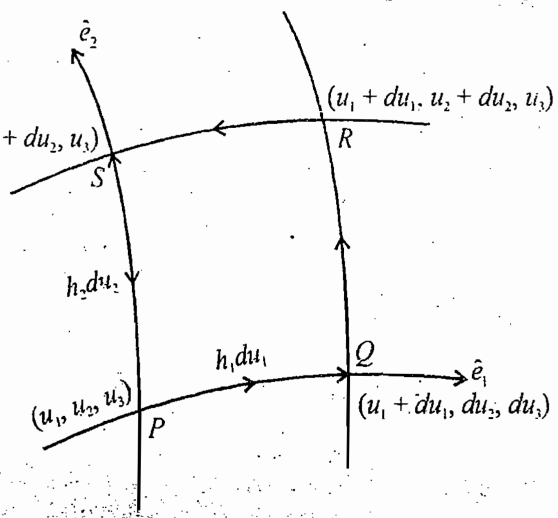


Figure 10.6

The unit normal vector to the surface dS is $\hat{e}_3$

$$\hat{n}\,dS = h_1h_2du_1du_2\hat{e}_3$$

Along PQ,

$$\vec{dl} = h_1du_1\hat{e}_1$$

$$\vec{A}\cdot\vec{dl} = A_{u_1}h_1du_1$$

To evaluate $\vec{A}\cdot\vec{dl}$ along RS, $h_1A_{u_1}$ has to be evaluated at $u_2 + du_2$.

For a derivable function $f(u)$

$$f(u+du) = f(u)+\frac{df}{du}du$$

Along RS

$$\vec{dl} = h_1du_1(-\hat{e}_1) = -h_1du_1\hat{e}_1$$

$$\vec{A}\cdot\vec{dl} = -\vec{A}\cdot\hat{e}_1h_1du_1 = -\left[h_1A_{u_1}+\frac{\partial}{\partial u_2}(h_1A_{u_1})du_2\right]du_1$$

So, $\vec{A}\cdot\vec{dl}$ along PQ & RS together gives $-\frac{\partial}{\partial u_2}(h_1A_{u_1})du_1du_2$.

Similarly, $\vec{A}\cdot\vec{dl}$ along QR & SP together gives $\frac{\partial}{\partial u_1}(h_2A_{u_2})du_1du_2$.

So, along $PQRS$

$$\oint\vec{A}\cdot\vec{dl} = \left[\frac{\partial}{\partial u_1}(h_2A_{u_2})-\frac{\partial}{\partial u_2}(h_1A_{u_1})\right]du_1du_2$$

$$= \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_{u_2})-\frac{\partial}{\partial u_2}(h_1A_{u_1})\right]dS$$

The coefficient of dS on right hand side gives the u_3 component of *curl* $\vec{A}$

$$(\nabla\times\vec{A})_{u_3} = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_{u_2})-\frac{\partial}{\partial u_2}(h_1A_{u_1})\right]$$

$$(\nabla\times\vec{A})_{u_1} = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3A_{u_3})-\frac{\partial}{\partial u_3}(h_2A_{u_2})\right]$$

$$(\nabla\times\vec{A})_{u_2} = \frac{1}{h_3h_1}\left[\frac{\partial}{\partial u_3}(h_1A_{u_1})-\frac{\partial}{\partial u_1}(h_3A_{u_3})\right]$$

So,
$$\nabla\times\vec{A} = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3A_{u_3})-\frac{\partial}{\partial u_3}(h_2A_{u_2})\right]\hat{e}_1+\frac{1}{h_3h_1}\left[\frac{\partial}{\partial u_3}(h_1A_{u_1})-\frac{\partial}{\partial u_1}(h_3A_{u_3})\right]\hat{e}_2$$
$$+\frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_{u_2})-\frac{\partial}{\partial u_2}(h_1A_{u_1})\right]\hat{e}_3$$

LAPLACIAN

Laplacian is divergence of gradient.

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right]$$

Proof:
$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\hat{e}_1+\frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\hat{e}_2+\frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\hat{e}_3$$

We know that

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_{u_1})+\frac{\partial}{\partial u_2}(h_1h_3A_{u_2})+\frac{\partial}{\partial u_3}(h_1h_2A_{u_3})\right]$$

Here, $\vec{A} = \nabla\phi$

So,
$$\nabla\cdot(\nabla\phi) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right]$$

PHYSICAL SIGNIFICANCE OF DIVERGENCE

The surface integral $\oint_S \vec{f}\cdot\hat{n}\,dS$ is called flux of $\vec{f}$ across the closed surface S. If $\vec{f}$ denotes the fluid velocity, then the flux of $\vec{f}$ across the closed surface denotes the quantity of fluid flowing across the closed surface S. If the flux is positive, there is net flow out of the surface and if the flux is negative, there is net flow into the surface.

The term divergence comes from interpreting $div\,\vec{f}$ as a measure of how much a vector field diverges from a point. The divergence of $\vec{f}$ can be defined with the help of Gauss Divergence theorem as

$$\nabla\cdot\vec{f}(x,y,z) = \lim_{V\to 0}\frac{1}{V}\oint \vec{f}\cdot\hat{n}\,dS$$

where V is the volume enclosed by the closed surface S around the point (x, y, z). In the limit $V\to 0$ means that we take smaller and smaller closed surface around (x, y, z) which means that the volumes they enclose are going to be zero. It can be shown that this limit is independent of the shapes of those surfaces.

The limit of ratio of flux through the surface to the volume enclosed by that surface gives a rough measure of the flow leaving a point.

PHYSICAL SIGNIFICANCE OF CURL

The line integral $\oint_C \vec{f}\cdot d\vec{r}$ is called circulation of $\vec{f}$ around C. So, the $curl\,\vec{f}$ is interpreted as a measure of circulation density. The can be understood by defining $curl\,\vec{f}$ with the help of Stoke's law as

$$curl\,\vec{f}(x, y, z) = \lim_{S\to 0}\frac{1}{S}\oint_C \vec{f}\cdot d\vec{r}$$

where S is the surface containing point (x, y, z) and bounded by a simple closed curve C. In the limit, the curve C, shrinks to a point (x, y, z) which causes surface S to have smaller and smaller surface area. The ratio of circulation to surface area in the limit makes $curl\,\vec{f}$ a rough measure of circulation density.

Now, Let us discuss by taking an example, how the curl of a vector field represents its rotational sense. Let us take an example of a vector field $\vec{f} = (1+x^2)\hat{j}$. The field lies in xy plane pointing y direction. The strength of field increases as we move away from y axis in either direction. i.e. positive & negative x.

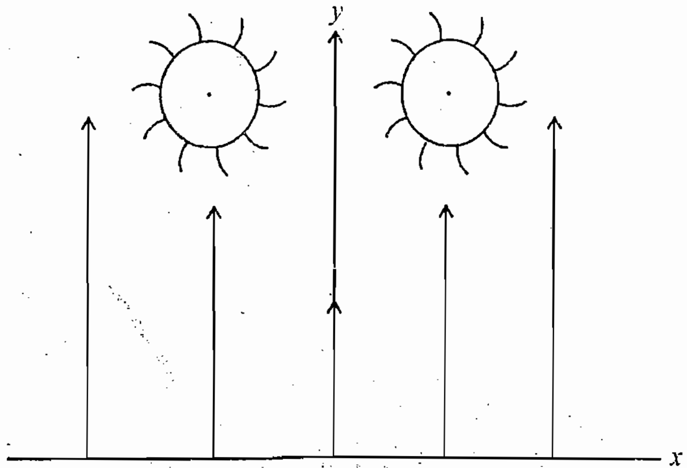


Figure 10.7

Let us suppose the field here denotes fluid velocity. If we put two light flywheel with paddles into water flow.

The wheel on the right of y axis will rotate in anticlockwise direction and the wheel on the left of y axis rotates in clockwise direction. In both the cases curl will be non zero and obeys right hand rule that is, the $curl\,f(x, y, z)$ points in the direction of thumb as you cup you right palm in the direction of rotation. So, the curl points towards positive z direction in the region right of y axis $(x > 0)$ and towards negative z direction in the region left of y axis $(x < 0)$.

Let us calculate $curl\,\vec{f}$

$$curl\,\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 1+x^2 & 0 \end{vmatrix} = 2x\hat{k}$$

$curl\,\vec{f} = 2x\hat{k}$ confirms what we have discussed.

i.e. $curl\,\vec{f}$ lies towards $\hat{k}$ for $x > 0$ and towards $-\hat{k}$ for $x < 0$.

If all the vectors had same direction and same magnitude then wheels won't rotate and hence there would be no curl, such fields are called irrotational meaning no rotation.

Now, let us write the expression of gradient, divergence, curl and Laplacian in cartesian, spherical polar and cylindrical coordinates.

(1) **Cartesian (x, y, z) :** Scalar function ϕ; vector field

$$\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

Gradient : $$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Divergence : $$\nabla\cdot\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Curl : $$\nabla\times\vec{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

Laplacian : $$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

(2) **Spherical (r, θ, ϕ) :** Scalar function T, vector field

$$\vec{f} = f_r\hat{e}_r + f_\theta\hat{e}_\theta + f_\phi\hat{e}_\phi$$

Gradient : $$\nabla\phi = \frac{\partial T}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{e}_\phi$$

Divergence : $$\nabla\cdot\vec{f} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, f_\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$$

Curl : $$\nabla\times\vec{f} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, f_\phi) - \frac{\partial f_\theta}{\partial\phi}\right]\hat{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial f_r}{\partial\phi} - \frac{\partial}{\partial r}(r f_\phi)\right]\hat{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r f_\theta) - \frac{\partial f_r}{\partial\theta}\right]\hat{e}_\phi$$

Laplacian : $$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

(3) **Cylindrical (ρ, ϕ, z) :** Scalar function T; Vector function

$$\vec{f} = f_\rho\hat{e}_\rho + f_\phi\hat{e}_\phi + f_z\hat{e}_z$$

Gradient : $$\nabla T = \frac{\partial T}{\partial\rho}\hat{e}_\rho + \frac{1}{\rho}\cdot\frac{\partial T}{\partial\phi}\hat{e}_\phi + \frac{\partial T}{\partial z}\hat{e}_z$$

Divergence : $$\nabla\cdot\vec{f} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\rho) + \frac{1}{\rho}\cdot\frac{\partial f_\phi}{\partial\phi} + \frac{\partial f_z}{\partial z}$$

Curl : $$\nabla\times\vec{f} = \left(\frac{1}{\rho}\frac{\partial f_z}{\partial\phi} - \frac{\partial f_\phi}{\partial z}\right)\hat{e}_\rho + \left(\frac{\partial f_\rho}{\partial z} - \frac{\partial f_z}{\partial\rho}\right)\hat{e}_\phi + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho f_\phi) - \frac{\partial f_\rho}{\partial\phi}\right)\hat{e}_z$$

Laplacian : $$\nabla^2 T = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial T}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2}$$

109. Let $\langle a_n \rangle_{n=0}^{\infty}$ and $\langle b_n \rangle_{n=0}^{\infty}$ be two sequence such that $\langle a_n \rangle_{n=0}^{\infty}$ and $\langle b_n \rangle_{n=0}^{\infty}$ converges respectively to A and AB, then $\langle b_n \rangle_{n=0}^{\infty}$ converges iff

(a) $A \neq 0$ (b) $A = 0$
(c) $B = 0$ (d) None of these

110. Let $a_n = \begin{cases} 1+\frac{1}{n}, & n \text{ is even} \\ -1-\frac{1}{n}, & n \text{ is odd} \end{cases}$. Then

(a) $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = 1$ (b) $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = -1$
(c) $\limsup_{n\to\infty} a_n = 1 \liminf_{n\to\infty} a_n = -1$ (d) $\limsup_{n\to\infty} a_n = 1 \liminf_{n\to\infty} a_n = 1$

111. The sequences $\{1, 0, 1, 0, 1, 0, ...\}$ is
(a) increasing sequence (b) decreasing sequence
(c) monotonic sequence (d) None of these

112. **Statement (A)** Every converges sequence is bounded.
Statement (B) Every bounded sequence is convergent.
(a) A is true, B is false (b) B is true, A is false
(c) A and B both true (d) A and B both false

113. Let $\{a_n\}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if

(a) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ exists (b) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+3}$ exists
(c) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ and $\lim_{n\to\infty} a_{3n}$ exists (d) None of these

114. If $(a_n)_{n=0}^{\infty}$ converges to a, for all n, $a \geq 0$, then $\{\sqrt{a_n}\}_{n=0}^{\infty}$ is

(a) converges to $\sqrt{a}$ (b) diverges to $\sqrt{a}$
(c) converges to a (d) diverges to a

115. If $\langle a_n \rangle_{n=1}^{\infty}$ is decreasing and bounded, then $\langle a_n \rangle_{n=1}^{\infty}$ is
(a) converges sequence (b) diverges sequence
(c) non-Cauchy sequence (d) None of these

116. Let a_n = least power of 2 that divides n. Then $< a_n >$ is
(a) diverges to infinity (b) bounded
(c) having a subsequences converging to 3 (d) converges

117. Let sequence $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ converges to A and B respectively, then $\langle a_n / b_n \rangle_{n=1}^{\infty}$ converges to A/B, if

(a) $b_n \neq 0$ for all n and $B = 0$

(b) $b_n \neq$ for some n

(c) $b_n \neq 0$ for all n and $B \neq 0$

(d) None of these

118. If $< a_n >$ is decreasing sequence of positive number $\sum a_n$ converges, then $\lim_{n\to\infty} na_n$ is

(a) ∞ (b) 0
(c) 1 (d) may not exist